THE EFFECTS OF CONCRETE TO SEMICONCRETE TO ABSTRACT INSTRUCTION ON THE ACQUISITION AND RETENTION OF FRACTION CONCEPTS AND SKILLS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1995

ACKNOWLEDGMENTS

The doctoral program and dissertation have been life-changing events, and my heart is filled with appreciation for those whose expertise and support have been instrumental in my professional and personal growth.

First, I thank my committee members, Don Bernard, Cynthia Griffin, David Miller, and Cary Reichard. Their feedback and collaboration greatly enhanced both the process and the product. Second, I would like to thank Mrs. Kathy French of St. John's County who created the fraction games upon which my study was based. Third, thanks go to Susan Houder and Margie Donnelly of P. K. Yonge Developmental School and Gail McEvoy of the Multidisciplinary Diagnostic and Training Program (MDTP) for their participation in the pilot studies. Fourth, I would like to thank Wade White and Yolanda Williams of South Ocala Elementary School, Jan McCall and Mary Kloman of Eighth Street Elementary School, David Ormond and Sheryl Robinson of Ward-Highlands Elementary School, and Gail Haselden and Connie Varee of Maplewood Elementary. These teachers and their school administrators welcomed me into their schools to conduct research and supported me with their time, expertise, and materials.

I also appreciate the contributions of Carolyn and Mike Harris, Cecil Mercer, and Susan Miller to the intervention. Sybil Brown was a blessing in research design and analysis (and multiple visits to the computer lab). Thanks go to Mary Sue Rennells and Mary Robertson Eisele for their help on the pretest. Thanks also go to Barbara Smerage for reducing my stress level significantly in her assistance with editing and typing the manuscript. I thank Elizabeth Gibbs, Cary Reichard, Suzanne Thomas, and Holly and Charlie Lane for their help in preparation for the defense.

Heartfelt thanks go to the membership of Delta Kappa Gamma Society International, Eta State, and Beta Alpha Chapter for their moral and financial support. The Gilbert-McNairy Scholarship (Eta State) and J. Maria Pierce International Scholarship helped to make my studies possible.

Now my list could go on forever, because describing the many acts of kindness from the family and friends during this process would require another dissertation. Special thanks go to my parents, Edward and Eula Mae Morgan Jordan; my sister, Amy; and my grandmother, Mary Lee. I draw on their strength, love, and prayers and am thankful for their continuing support. I am especially indebted to my students in Farmville who taught me much about strength and tenacity. I also appreciate the love of my aunts, uncles, cousins, and friends "back home."

Faye Dozier, Lynn Hobbs, Debbie Metcalf, Dot Spruill, and Edith Warren of Sam D. Bundy Elementary School and faculty in the School of Education at East Carolina University have supported me faithfully in this process. Special thanks go to Dr. Betty Levey for her wisdom and her encouragement to come to Florida for this adventure.

The Department of Special Education faculty and staff at the University of Florida have provided wisdom, knowledge, laughter, and friendship. The MDTP staff have made my association with that project a pleasant learning experience.

I am certain that my fellow doctoral students in our department are the finest in the land, and their friendship and quality of character have influenced me in the happiest of ways. They are my confidants and my clowns, and I am so glad God has seen fit for our paths to cross. I am very thankful for my Florida family, Holly and Charlie Lane and David and Margaret Allsopp, whose friendship and humor I treasure. I appreciate my roommate, Connie Daughtry, for her patience with my piles of computer printouts and articles.

I cannot imagine this experience without my mentor, Cecil Mercer. He has been my role model and inspiration, a scholar, a teacher, and a friend. The great thing about Cecil is that he has wisdom, compassion, and humor in nearly perfect amounts. To be able to work with him is a terrific blessing. Ann Mercer is appreciated as well, for it is her voice that often reassured me that a doctoral program is indeed survivable.

Finally, I thank my Lord for extra strength and courage when I needed it most and for ministering to me through wonderful people and events. His guiding presence in my life has made this endeavor possible and, in fact, a large time.

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Abstract of Dissertation Presented to the Graduate School of
the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

THE EFFECTS OF CONCRETE TO SEMICONCRETE TO ABSTRACT INSTRUCTION ON THE ACQUISITION AND RETENTION OF FRACTION CONCEPTS AND SKILLS

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May 1995

Chairperson: Cecil D. Mercer
Major Department: Special Education

The purpose of this investigation was to compare the effects of concrete to semiconcrete to abstract (CSA) instruction in fraction concepts and skills to the effects of instruction that was based on textbook curriculum. Effects were measured across time in three posttests given immediately after instruction, one week after instruction, and as late as possible in the school year. The subjects in this study were fourth-grade students in four elementary schools. The classrooms included students with disabilities and students at-risk for academic failure.

The procedure for this study consisted of five phases including two pilot studies, a training inservice for the teachers of the treatment groups, implementation, and posttreatment. Treatment and control groups were determined to have no preexisting differences. A repeated measures analysis of variance (ANOVA) with one between subject factor (treatment versus control) and one within subject factor (performance over time) was computed to determine whether any significant differences were present among the levels of the experimental treatments. Due to interaction effects in posttests 2 and 3, follow-up univariate analyses of variance were computed to determine effects in each of the treatment groups. Performance of both groups changed significantly from pretest to posttest 1. No significant differences were found in either group from posttest 1 to posttest 2 and from posttest 2 to posttest 3. At each posttest measure, treatment and control groups differed significantly in achievement in fraction concepts and skills. Students who were taught fraction concepts and skills in a CSA sequence performed significantly better on each of the three posttest measures than students taught with textbook curriculum. The results of this investigation provide practical implications for classroom instruction and future research.

CHAPTER I
INTRODUCTION TO THE PROBLEM

The purpose of Chapter I is to present issues critical to the present study and provide a rationale for the study. This section discusses issues relating to mathematics performance and curriculum and focuses specifically on students with learning disabilities.

Mathematics Performance and Curriculum

Mathematics performance and achievement have been areas of concern and discussion for many years. In the early years of education, mathematics was seen as mental exercise. Today, performance in mathematics influences career opportunities for students.

Yates and Yates (1990) describe the challenge of mathematics learning. They state, "The acquisition of knowledge is a remarkably slow process involving hours of practice on component skills and, as in the case of mathematics, on procedural operations involving scores of decision elements" (p. 226).

A variety of factors influence math performance and the goals of math education. Two of the most prominent factors are standards and mathematics curriculum.

Standards and Expectations for Learning

The National Council of Teachers of Mathematics (NCTM) have developed a set of standards for mathematics learning that they hope will provide students with increased mathematical understanding and power to use mathematics (Romberg, 1993). The standards encourage mathematics professionals to provide opportunities for students to reason and problem solve as well as build self-concept about their ability to do math (NCTM, 1989). Romberg (1993) described the standards as a rallying flag for math educators. In contrast, Hofmeister (1993) cautioned that the standards lack a research base and may, in fact, be detrimental to the learning of many students, especially learners with disabilities.

Engelmann, Carnine, and Steeley (1991) listed the goals for students according to NCTM's standards as (a) to value mathematics, (b) to reason mathematically, (c) to communicate mathematics, (d) to problem solve, and (e) to instill confidence. The authors noted, however, that only the goal of building confidence has been attained by current mathematics instruction, as American students view their competency level as high when their test scores in comparison to other countries are low. Mtetwa and Garofalo (1989) presented a contrasting opinion of students' levels of math confidence. The authors described myths of mathematics that are readily believed by students. Those myths include (a) students must automatically apply an algorithm to problems; (b) mathematics is a loose collection of facts, rules, formulas, and procedures; (c) there is only one right answer; and (d) only geniuses can create mathematics.

Mathematics Curriculum

The spiral curriculum according to Engelmann et al. (1991) provides coverage of many topics and in-depth teaching of none. The practicality of the spiral curriculum falls far from the curriculum's intent, which is to add depth with each year of instruction. A review of textbooks (e.g. *Silver Burdett & Ginn Mathematics, Mathematics Today, Heath Mathematics, Mathematics in Action*) reveals that objectives are addressed and reviewed in a yearly cycle of instruction. More difficult objectives are added with each successive year.

Mathematics Performance and Students with Learning Disabilities

Students with learning disabilities in math consistently lag behind their nondisabled peers in achievement. Mastropieri, Scruggs, and Shiah (1991) stated that mathematics deficits present serious problems for students with learning disabilities. Problems that begin in elementary grades continue throughout secondary grades (Cawley & Miller, 1989). The researchers reported that students with learning disabilities progress one year for every two years that regular learners progress. On a positive note, Cawley and Miller (1989) added that although progress is slow, students with learning disabilities *do not stop learning*.

Specific learning difficulties contribute to poor math performance. These difficulties include deficits of attention and memory (Zentall & Ferkis, 1993), visual-spatial abilities (Garnett, 1992), transfer (Brownell, Mellard, & Deshler, 1993) and information processing (Lenz, 1992). In addition, students who do have learning strategies may misuse them (Montague, 1992).

Mercer, Jordan, and Miller (1994) assert that NCTM's high standards promote high levels of frustration in both teachers and students who have difficulty attaining objectives of the traditional curriculum. With new, tougher math standards a certainty and the poor math performance of students with disabilities, the impetus for improved math instruction is great.

Statement of the Problem

The problem investigated in this study was the effectiveness of teaching students fraction concepts and skills through a concrete to semiconcrete to abstract sequence of instruction compared to teaching similar skills with a textbook curriculum. The experimental questions were as follows:

1. Is the concrete to semiconcrete to abstract sequence of instruction more effective than a

textbook curriculum in teaching students to acquire fraction concepts and skills?

2. Is the concrete to semiconcrete to abstract sequence of instruction more effective than a textbook curriculum in teaching students to maintain fraction concepts and skills for one week?
3. Is the concrete to semiconcrete to abstract sequence of instruction more effective than a textbook curriculum in teaching students to retain fraction concepts and skills for at least two weeks?

The problem examined in this study is important for several reasons. First, the study expands the current research data on teaching mathematical skills using the concrete to semiconcrete to abstract (CSA) sequence of instruction. Second, this study provides practitioners with information about skill retention following two types of instruction: (a) textbook-based and (b) CSA. Third, the study provides information about the promotion of conceptual and practical understanding of mathematics in learners.

Rationale of the Study

As stated previously, improvements in mathematics instruction are necessary to assist learners, including learners with disabilities, to (a) meet the challenges presented by present curriculum and (b) cope with learning difficulties that may be present. In the area of fractions, instruction often focuses on abstract application of multiplication and division skills, without emphasis on concepts of equivalence, addition and subtraction.

Early learning theory (Bruner, 1966; Dienes, 1964; Gagne, 1977; Piaget, 1960; Vygotsky, 1978) underscores the importance of initial concrete experience and appropriate tasks to build conceptual understanding. As an extension of the learning theory, Underhill (1981) applied the concrete to abstract sequence of instruction to mathematics learning.

While research has shown that direct instruction using the concrete to semiconcrete to abstract sequence can be effective in teaching mathematics concepts and skills, most of these studies have been limited to small groups of students. Fraction learning and instruction research has also been conducted primarily with small groups, and while concrete instruction has been implemented frequently, neither a sequence of presentation including semiconcrete and abstract phases nor direct instruction in concepts have been included.

The present study combines these areas of research and uses the concrete to semiconcrete to abstract sequence of instruction to teach fraction concepts and skills in large group settings. Information from the study will add to the existing data base on mathematics instruction. The findings from a systematic evaluation of instruction can also inform inservice and preservice teachers in effective teaching techniques. Students with and without disabilities are likely to benefit from increased knowledge concerning effective teaching practices.

Definition of Terms

Terms used in the study and their definitions are presented in the following section. Terms chosen for inclusion in this section are critical to understanding implementation procedures and observed results.

Concrete learning experience. Underhill et al. (1980) described a concrete learning experience as "one that helps the learner relate manipulative processes and computational processes. The learner focuses on both learner-manipulated objects and the symbolic processes which accompany, correlate to, and describe the manipulations" (p. 28).

Semiconcrete learning experience. Underhill et al. (1980) described a semiconcrete learning experience as the following:

> A semiconcrete learning experience is one that is visual, but is not tactile or kinesthetic as is the concrete learning experience. Examples include drawing pictures or diagrams; watching slides, filmstrips, or movies; watching a demonstration with manipulatives; and looking at pictures and diagrams in textbooks or worksheets. The focus is on building associations between visual modeling and symbolic processes. (p. 29)

Abstract learning experience. "An abstract learning experience is one in which the learner uses no manipulatives or visual process reinforcers to build associations"

(Underhill et al., 1980, p. 30). Only written symbols such as numbers are used.

Continuous quantities. Continuous quantities when describing fractions are parts of a region or area.

Criterion. The criterion is a specific value that must be obtained during instruction to indicate mastery.

Denominator. The denominator is the part of a fraction that indicates the number of pieces in the whole or the entire set. It is also the number below the fraction bar in a written symbol for a fraction.

Discrete quantities. Discrete quantities are amounts that can be counted.

Equivalent fractions. Equivalent fractions represent the same number.

Fraction. A fraction is a number that identifies part of a group or region.

Like fractions. Like fractions have the same denominator.

Maintenance. Maintenance is an instructional procedure in which periodic checking occurs to ensure the skill is still performed at criterion (Peterson, 1987/88, p. 9). In this study maintenance is measured one week after instruction.

Mixed number. A mixed number is a number that includes a whole number and a fraction.

Numerator. The numerator is the part of the whole that is in use. It is also the number above the fraction bar in a written symbol for a fraction.

Retention. Retention is the ability to recall information learned previously. Retention is measured three weeks after instruction in this study.

Delimitations of the Study

The study is delimited by geographical restrictions to Marion County, Florida, a medium-sized county located in the north central part of the state. Subjects were all fourth-grade students in public school.

Limitations of the Study

Because this study included only elementary students, the findings should not be generalized to middle or high school students. In addition, the findings cannot be generalized to other mathematics skills. One should exercise caution in generalizing results to populations outside of Marion County.

Summary

Research in math instruction that teaches concepts and develops abstract thought is clearly needed in the area of fractions. The intent of this study was to advance the knowledge of effective instruction in fractions using a concrete to semiconcrete to abstract approach to instruction. Acquisition of fraction concepts and skills, maintenance, and retention were examined specifically. Results of this study have implications for teachers of mathematics.

Chapter II presents a review of literature relevant to this study. Methodology used in implementation is described in Chapter III. Results are discussed in Chapter IV, and implications are presented in Chapter V.

CHAPTER II
REVIEW OF RELATED LITERATURE

The purpose of this chapter is to review, analyze, and synthesize the professional literature related to the teaching of fractions to elementary and middle school students. Specific areas relevant to fraction instruction include (a) children's cognitive approaches to fractions; (b) concrete, representational, or abstract instruction in mathematics; and (c) the use of games in mathematics instruction.

The chapter is divided into 10 sections. First, criteria are described for selection and inclusion of studies. Second, issues related to fraction instruction within the mathematics curriculum are discussed. Third, student characteristics related to the learning of fractions are described. Fourth, the relationship of fraction instruction and the learning of fractions to the student with mild disabilities is discussed. Fifth, implications for fraction instruction are presented. The sixth, seventh, and eighth sections of the chapter present research related to fractions. In the ninth section, research is presented on concrete, representational (semiconcrete), or abstract instruction. Finally, research relevant to the use of games in mathematics instruction is presented.

Criteria for Selection of Relevant Literature

To be considered for the literature review, studies were required to meet the following criteria.

1. Research questions addressed (a) children's cognitive approaches to fractions; (b) concrete, representational, or abstract instruction in mathematics; and (c) the use of games in mathematics instruction.

2. Mathematics performance was the dependent measure in the studies. Some studies used interviews to assess students' performance; other studies used tests.

3. Elementary or middle school students were the subjects in the studies.

4. Studies were data based, published, and included adequate descriptions of subjects, procedures, and results.

Studies were considered for inclusion in the literature review if they met all criteria and were conducted between 1980 and 1993. In addition, important studies conducted before 1980, doctoral dissertations, and graduate studies conducted through the Department of Special Education at the University of Florida were examined for relevant findings.

A total of 29 studies were located that met the inclusion criteria, and 18 studies were adapted from a literature review by Harris (1992). Research on students' approaches to fractions was included in nine studies. Research from the Rational Number Project (Behr, Wachsmuth, & Post, 1985), a federally funded program to study rational numbers was included in six studies. Two studies included research on fractions using concrete models. Seven studies addressed concrete, representational, or abstract instruction. Use of instructional games was addressed in three studies.

The Teaching and Learning of Fractions

Math experts agree that fractions pose problems for teachers and students. While this consensus exists, few agree upon what to do to improve the teaching and learning of fractions (Good, Mulryan, & McCaslin, 1992; Hope and Owens, 1987). This discussion of fractions in mathematics includes a description of present fraction instruction in elementary classrooms. Learner characteristics that influence understanding of fractions are also considered. Implications for the student with mild disabilities are discussed. Finally, implications from the research for improvements in fraction instruction are presented.

Present Fraction Instruction

Silbert, Carnine, and Stein (1990) asserted that the topic of fractions is one of the most complex sets of skills covered in elementary mathematics. Many component skills are involved in fractions, and mastery of a set of skills in computation often depends on mastery of background skills (i.e., fraction identification and equivalence).

Present fraction instruction is characterized by a lack of understanding, strict adherence to procedures, and a misapplication of rules because students do not understand reasons for the rules. Baroody and Hume (1991) stated that much of fraction instruction is meaningless. Baker, Carpenter, Fennema, and Franke (1994) and Bezuk and Bieck (1993) stated that students lack understanding of fractions because instruction on fraction concepts is superficial at best and does not provide opportunities to develop concrete understanding. In support of this view, Baroody and Hume (1991) asserted that much of instruction in fractions is not developmentally appropriate. Moreover, Silbert, Carnine, and Stein (1990) asserted that lack of structured teaching and lack of adequate practice for mastery cause problems for students learning fractions. Instruction fails to show students the connections between what they know already and what they need to learn about fractions (Hiebert, 1985).

Deterrents to Adequate Instruction

Competing factors exist which may preclude adequate instruction in fraction concepts. Time for fraction instruction is limited by time taken for other mathematical topics. Bezuk (1988) asserted that not enough time is devoted to fraction instruction. She further asserted that manipulative objects are not used to maximum benefit and that abstraction of concepts begins too soon. Nik Pa (1989) admonished that fraction instruction often focuses on computation at the expense of understanding.

Engelmann et al. (1991) stated that textbooks fail to address fractions properly in three important ways. First, the fraction concept may only be addressed for a few weeks. Second, corrective procedures are relatively nonexistent. Third, there is too little practice with concrete examples. Baker et al. (1994) stated that textbooks emphasize part-whole relationships of fractions, thus ignoring other areas such as ratio and division.

The spiral curriculum is intended to increase difficulty for older topics over the years as new topics are introduced on a basic level (Underhill, 1981). However, Engelmann et al. (1991) asserted that the spiraling curriculum actually provides inadequate attention to concepts and encourages little mastery of skills. The nature of the spiral curriculum of mathematics relegates attention to mastery of fraction concepts to a relatively low priority. Students are exposed to fractions on a yearly basis and are provided with algorithms and procedures for getting the computation right. This glossing over of concept instruction prevents many students from mastering a concept well enough to apply it.

Learner Factors Involved in Fraction Understanding

Learning of fractions requires student input as well as teacher or textbook instruction. Fractions place heavy demands on student attention and memory (Baroody & Hume, 1991). For example, when students rename fractions to add or subtract, they must remember the original fractions while they apply multiplication or division concepts to two or more numbers.

Students may not see that fractions are relationships and that fractions change with different contexts and applications. Baroody and Hume (1991) stated that students may have incomplete or inefficient fractions strategies. The authors place students' difficulties with fractions into three categories: (a) definition of the whole, (b) equal partitioning, and (c) comparisons. Problems with defining the whole are attributed to a lack of examples of both continuous and discrete quantities. Early references to "half" that represent quantities that are not exact may contribute to misunderstanding of partitioning. Students' understanding of whole numbers interferes with comparisons of fractions in examples where larger denominators mean smaller quantities.

The multiple meanings of fractions may cause students confusion when working with fractions in various contexts (Behr, Wachsmuth, Post, & Lesh, 1984; Hiebert, 1985). Hiebert (1985) stated that a fraction meanings include a point on a number line, a part of a whole, a ratio, and a part of a set. He asserted that students do not internalize all of these meanings. Moreover, students memorize procedures by rote without comprehending why the procedures work. Without the needed understanding, students overlook inconsistencies and errors in their work when they eventually misapply the rules. A lack of proficiency in basic number facts may also contribute to difficulties with fractions (Hope & Owens, 1987).

Informal knowledge has been an area of popular research in fractions in recent years. Researchers such as Baker et al. (1994), Mack (1990), and Hunting and Sharpley (1988), proposed that students bring a store of informal knowledge and assumptions about fractions to the classroom. Informal knowledge comes from daily childhood experiences of sharing and other activities where dividing quantities into parts is required. Sometimes informal knowledge may cause students to construct strong misconceptions about fractional numbers or take rules that they know about whole numbers and apply them incorrectly to fractions. For example, a large number in the denominator indicates a small fraction. Students often apply their knowledge of whole numbers and incorrectly assume that as the denominator increases, so does the fraction.

Fraction Understanding and Students With Mild Learning Disabilities

Students with mild to moderate learning disabilities are especially hurt by meaningless instruction (Baroody & Hume, 1991). In addition to problems with instruction, students with mild to moderate learning disabilities have learning characteristics that may amplify their difficulties with fractions. Bley and Thornton (1981) summarized visual and auditory deficits related to learning disabilities and related the deficits to disabilities in math. The implications of these areas for fractions are described in Table 1.

As shown in Table 1, students with learning disabilities may have problems in a variety of areas that influence their achievement in fractions. Attention must be given to these learning differences in evaluating the effectiveness of fraction instruction in inclusive classrooms.

Implications for Fraction Instruction

The literature is replete with recommendations regarding fraction instruction, and many of the recommendations are made for students with learning disabilities (e.g., Rivera & Bryant, 1992). Baroody and Hume (1991) stated that instruction in fractions needs to focus on understanding and should build on informal knowledge. Symbolic knowledge should be linked to concrete knowledge. The authors further asserted that students should be actively involved in instruction and teachers should facilitate learning with questions.

Use of Manipulative Objects

Baroody and Hume (1991), Mercer and Miller (1992b), and Baker et al. (1994) recommended that a variety of manipulative objects be used in different contexts in instruction. In contrast, Behr, Wachsmuth, and Post (1988) stated that research using manipulative objects has been equivocal.

Table 1

Common Difficulties that Affect Performance in Fractions with Students with Learning Disabilities

Learning Disability		Performance in Math
Visual Perception	Figure-ground	- has difficulty reading digits in the form of a fraction or mixed number
	Discrimination	- has difficulty understanding arrangement of digits and fraction bar
	Spatial	- has difficulty writing fraction symbols - has difficulty using pictorial representations of fractions - has difficulty aligning digits to write fractions
Auditory Perception		- has difficulty writing dictated fractions or mixed numbers - has difficulty with oral word problems
Motor Skills		- has difficulty aligning digits to write fractions
Memory	Short-term	- has difficulty remembering fraction equivalence - has difficulty remembering fraction rules - cannot remember symbol meaning
	Long-term	- works slowly to master fraction equivalence facts - performs poorly on review activities - forgets steps in algorithms
	Sequential	- has difficulty with multistep problems such as repeated fraction reduction - has difficulty with multistep word problems
Language	Receptive	- has difficulty relating arithmetic terms to meaning (e.g., numerator, denominator, mixed number) - has difficulty with the multiple meanings of fractions
	Expressive	- does not use the vocabulary of arithmetic - has difficulty verbalizing steps in problem solving

Table 1--continued.

Reading		- is unable to comprehend information from word problems
Abstract Reasoning		- has difficulty solving word problems - cannot make comparisons of size and quantity
Meta-cognition		- has difficulty in selection of appropriate strategies for problem solving - is not aware of reasonableness of answers - cannot generalize strategies to other situations
Social and Emotional Factors	Impulsive	- makes careless mistakes in fraction computation or equivalence - does not attend to detail in problem solving (e.g., fractions that need to be reduced
	Short attention span	- does not complete work - has problems with multistep problems - is off- task during instruction
	Passivity	- omits problems - appears disinterested - lacks strategies for problem solving
	Self-esteem	- lacks confidence - gives up easily

Note. Adapted from Mercer (1991). Used with permission.

Fraction Concepts

Hiebert (1985) asserted that the more compelling need of fraction instruction is to solidify the knowledge students have rather than just add to students' knowledge. For example, students need to have enough sense of what answers should be to problems through estimation to be able to check whether their answers are reasonable. Baker et al. (1994)

stated that fractions should not be presented as new concepts but as new uses for concepts already learned. Teachers need to provide the connections for students by extending aspects of whole numbers that apply to fractions (Baker et al., 1994; Clements & Del Campo, 1989). For example, both whole numbers and fractions can be used to represent quantities.

Both Baroody and Hume (1991) and Baker et al. (1994) recommend that teachers wait to introduce fraction symbols until students are familiar with the fraction concept. It is students' readiness to learn fractions and their thinking about fractions that are investigated in the studies in the first section of related literature.

Studies Investigating Students' Approaches to Fractions

Nine studies were located between 1976 and 1992 that investigated students' approaches to fractions. The studies are analyzed and described in terms of subject characteristics, experimental procedures, research designs and measurement methods, maintenance and generalization measures, and results. Elements of the studies that are considered relevant are discussed in detail.

Description of Studies

The purpose in the nine studies summarized in Table 2 was to investigate the thinking of students as they solve tasks which involve fractions. Novillis (1976) determined the ability of 279 students to solve fraction tasks within

Table 2

Review of Studies Investigating Students' Approaches to Fractions

Author(s)	Subjects	Treatment	Measure
Novillis (1976)	277 fourth and fifth graders	test	*Fraction Concept Test* to measure students ability to solve tasks at each level of fraction hierarchy

Purpose: To investigate whether students who achieve concepts on the hierarchy of skills have also mastered subconcepts of the skill.

Results: In some cases students had mastered subconcepts but not in others. The subconcepts of part-whole noncongruent parts was found to be more difficult than part-group, noncongruent parts. Limitations: The hierarchy used in the study was incomplete. Students' skill profiles were diverse, which made the samples too small to study some of the more difficult concepts.

Table 2--continued.

Author(s)	Subjects	Treatment	Measure
Larson (1980)	382 students seventh graders placed in three tracks by ability	test	pre and post-test *Locating Fractions on the Number Line* to test number line tasks where equivalence and length varied

Purpose: To examine seventh graders' performance on various fraction and number line tasks where length and equivalence changed.

Results: Students found it easier to find points on number lines of one unit. Also, students could more easily find fractions on number lines when the units were divided into the same number of parts as the denominator of the fraction.

Author(s)	Subjects	Treatment	Measure
Peck & Jencks (1981)	20 students sixth graders	interviews where students responded to questions by using drawings or physical materials to show their thinking in multiplying or using fractions	interview responses

Purpose: To examine students' concepts of fractions.

Results: Nineteen of the 20 students could show the multiplication. Eleven of these students sketched fractions incorrectly, causing them to be unable to compare correctly. Seven of the same 11 used portions of rules in the wrong situation. Nine students sketched fractions correctly, one of those students could not use it to compare or add fractions. Five students could compare correctly. Two of the five students who could compare could add fractions and represent the fractions with a drawing. The other three could add using a common denominator rule, but could not represent the fraction.

Table 2--continued.

Author(s)	Subjects	Treatment	Measure
Bright & Harvey (1984)	41 students eighth graders	tests on equivalent fractions,	2 ten item tests where the student supplies a missing numerator or denominator

Purpose: To investigate students' abilities to find equivalent fractions.

Results: Three error patterns appeared in the test data. In one pattern, students maintained the difference between numerator and denominator. For example, in the problem 8/12 = 2/?, the answer would be 6, which is a difference of 4. In another pattern, students divided one numerator by the other numerator to get the missing denominator. The answer to 8/12 = 2/? would be 4. A third error pattern resulted from students' use of multiple strategies. For example, 8/12 = 2/? would result in 48 in the denominator because 8 divided by 2 is 4 and 4 multiplied by 12 is 48. Students changed strategies for finding equivalent fractions between the two tests. When the denominator was missing, the division strategy was used more often.

Pothier & Sawada (1983)	43 students 8 in kindergarten 8 Grade 1 12 Grade 2 15 Grade 3	five partitioning tasks related to cutting a cake, used sticks to represent	observation and interview

Purpose: To investigate partitioning as a process in the construction of rational number concepts.

Results: Researchers used data analysis to develop a theory to explain the partitioning process. Students first learn to partition into two parts, then into powers of two. Next, students learn to partition into even numbered parts. To partition into an odd number of parts, they must avoid beginning with a median cut.

Table 2--continued.

Author(s)	Subjects	Treatment	Measure
Vance (1986)	131 students 62 sixth gr. 69 seventh gr follow- up test with 12 sixth and seventh graders group was divided into low, average, and high ability	two- part test on ordering decimals and common fractions follow- up test given individually with interview	two- part test Part 1- find largest and smallest of three numbers Part 2- write a number between two numbers

Purpose: To assess the ability of sixth and seventh graders in put rational numbers in order.

Results: Seventh-grade students did better than sixth grade students overall. The greatest difference between the groups was on Part 1 (64% average score for 7th grade; 54% average score for 6th grade).

Author(s)	Subjects	Treatment	Measure
Cuneo (1987)	37 seventh graders 32 college students	computation task and 2 estimation tasks Tasks lasted 30- 60 minutes.	assessment of tasks

Purpose: To report ability to perform fraction tasks.

Results: On the computation tasks, 19 seventh graders were assessed to have "good" ability; 18 seventh graders were assessed to have "poor" ability. Eighteen college students were assessed to have "good" ability and 14 college students were assessed to have "poor" ability. Subjects who did poorly on estimation of fraction size also did poorly on computation tasks. However, poor performance on computation tasks did not guarantee a lack of understanding of fraction size.

Table 2--continued.

Author(s)	Subjects	Treatment	Measure
Mack (1990)	8 sixth graders of average ability	individual instruction in addition and subtraction of fractions in a one- to-one setting for six weeks	assessment tasks designed by the researcher

Purpose: To investigate the development of fraction understanding from two perspectives: (a) students' ability to build on informal knowledge and (b) effect of knowledge of procedures on students' ability to extend informal knowledge.

Results: Students built on informal knowledge when they could link symbols with real-world situations, but their ability to link fractions and symbols was initially weak. Moreover, the students' ability to use rote procedures with whole numbers often interfered with their ability to solve fraction problems. For example, students found it difficult to compare one-sixth and one-eighth correctly because of their knowledge that eight is more than six. Transfer of knowledge from one problem solving situation to another was limited. For example, students would respond that four-fifths and five-sixths were equal because each had one piece missing. Discrete models posed more problems than continuous models.

Table 2--continued.

Author(s)	Subjects	Treatment	Measure
Singer & Resnick (1992)	15 students 5 sixth graders 8 seventh graders 2 eighth graders 4 served in Chapter 1	15 probability problems based on proportional reasoning Students were asked to think aloud.	assessment interviews and performance on tasks

Purpose: To examine the representations used by children in their reasoning about probabilities before proportional reasoning has been mastered.

Results: Students preferred a part-part representation (e.g. comparison of the number of boys in a classroom to the number of girls) to solve problems of proportion rather than part-whole representations. Interactions were found to be significant between information type (complete, partial-whole, and partial-part) and quantity type (equal and unequal amounts of objects).

the *Hierarchy of Selected Subconcepts of the Fraction Concept (HSSFC)* and the relationship of achievement in some of the tasks in the hierarchy to other tasks in the hierarchy. Larson (1980) investigated students' strategies for finding points on a number line. Peck and Jencks (1981) followed students' progression through a sequence of multiplication and fraction tasks. Bright and Harvey (1984) researched the change in strategies students used for finding equivalence between a pretest and a posttest. Pothier and Sawada (1983) observed and interviewed students on their approach to a cake-cutting task. Vance (1986) tested whether sixth and seventh graders could compare fractions and then asked the students in a follow-up study to explain their answers. Cuneo (1987) measured students' ability to solve computation and estimation tasks using fractions. Mack (1990) studied eighth graders' abilities to solve novel tasks. Singer and Resnick (1992) assessed students' ability to solve probability problems (ratio proportion) and asked students to explain their work.

Subject Characteristics

Most of the studies reported included students without disabilities. Some studies, however (Mack, 1990; Singer & Resnick, 1992), include students who would be considered less able in mathematics or fractions. Mack's studies included students described as having difficulty understanding the concept of fractions. Singer and Resnick's study included 5 sixth graders, 8 seventh graders, and 2 eighth graders at a Catholic parochial school serving primarily a minority population. Four of these students received services through Chapter 1.

One study included only elementary students. Pothier and Sawada (1983) included 43 students in kindergarten through third grade in their study. Six studies (Bright & Harvey, 1984; Larson, 1980; Mack, 1990; Peck & Jencks, 1981; Singer & Resnick, 1992; Vance, 1986 included middle school students (i.e., grades 6, 7, and 8). Larson's (1980) study included 382 seventh graders. Bright and Harvey's (1984) study included 41 eighth graders. Peck and Jencks (1981) reported the responses of a group of 20 students which were representative of the 60 sixth graders at their school. Vance's study included 131 students in seventh and eighth grades. Mack's studies included 16 sixth-grade students. Singer and Resnick (1992) included 15 students in sixth, seventh, and eighth grades.

Novillis (1976) included both elementary and middle school students. Novillis' study included 279 fourth, fifth, and sixth graders in self-contained classrooms.

One study included 32 adult college students and 37 seventh graders (Cuneo, 1987). The students were recruited through newspaper ads and flyers.

Experimental Procedures

Random sampling was used in only one of the studies (Peck & Jencks, 1981). The researchers reported that the responses of their sample of 20 students reflected the general responses given by the group of 60 students with whom they had worked at the school. The seventh graders in Cuneo's (1987) study were recruited by the investigator to take part in the study.

Other studies used whole classroom groups (Bright & Harvey, 1984; Novillis, 1976; Vance, 1986) or individual students (Mack, 1990; Singer & Resnick, 1992). Mack's study included eight sixth graders, chosen for their need to learn about fractions. No explanation is provided for the choice of specific students from grades 6, 7, and 8 in the study by Singer and Resnick (1992). The researchers did state, however, that they selected their grade levels to reflect the age of transition to full proportional reasoning, which they refer to as a Piagetian concept.

In two studies (Larson, 1980; Novillis, 1976), the classroom grouping was a result of assignment to academic tracks or groups. The studies included students who were placed in three tracks for math instruction.

Pothier and Sawada (1983) used a method of theoretical sampling that allowed the researchers to gather their sample during the intervention. Students were chosen to participate because the interviewer believed that they would feel free to respond during the clinical session.

Finally, some studies assigned an ability level to students based on pretest performance. After Vance's initial study of 129 sixth and seventh graders, he gave the same test in an interview format to another 12 sixth and seventh graders, including one boy and one girl who performed at each achievement level according to their teachers: high, average, and low.

Implementation. Interviews with students were used in two studies. Peck and Jencks (1981) conducted interviews where students responded to questions by using drawings or physical materials to show their thinking. Students were asked first to develop a picture of multiplication equations using graph paper. A discussion of fractions followed, and students were then asked to draw fractions and show comparison and addition of fractions with unlike denominators through sketches.

Pothier and Sawada (1983) observed and interviewed students to determine their approach to a partitioning task. The researchers asked students to complete five partitioning tasks related to cutting a cake. Students used sticks to show how they would cut a cake into the requested number of equal parts. Younger students were asked to cut a cake for a relatively small group of people (two, four, three, and five). Older students were asked to cut a cake for larger groups that were multiples of three, four, and five.

In four studies the researchers simply studied student responses on tests. Novillis (1976) constructed the Fraction Concept Test to determine whether students had mastered subconcepts related to broader concepts within a hierarchy of skills that she had developed (Hierarchy of Selected Subconcepts of the Fraction Concept). According to the hierarchy, if students could associate three-fourths with six of eight objects shaded, then they would also be expected to associate three-fourths with three of four objects shaded.

Bright and Harvey (1984) used two 10-item tests that required the student to supply a missing numerator or denominator. The researchers noted the strategies used by individual students to find equivalent fractions. Cuneo (1987) used one test of computation and two tests of estimation of fraction size.

Larson (1980) administered a pretest and posttest that she designed, Locating Fractions on the Number Line. This test consisted of 16 multiple-choice items to test ability to use the number line where equivalence and length varied.

Other studies used both testing and interviews to gather information on students' approaches to the solution of fraction problems. Vance (1986) used a two-part test. The first part of the test assessed whether students could find the largest and smallest of three fractions or three decimals; the second part of the test assessed whether students could write a fraction between two other fractions and a decimal between two other decimals. In follow-up interviews, Vance gave the same test to individuals and asked them to explain their answers.

Students in the study by Singer and Resnick (1992) solved 15 probability problems based on proportional reasoning. Students were asked to think aloud to solve the problems. The investigators modeled a think-aloud procedure for the students before initial assessment. Assessment interviews were conducted in conjunction with the tasks.

Mack (1990) used individualized instruction in addition and subtraction of fractions. The approach to instruction was one of problem solving where the teacher and student were partners. Assessment tasks were used to measure student learning.

Materials. Concrete materials were used in three studies (Mack, 1990; Peck and Jencks, 1981; Pothier & Sawada, 1983). Other studies provided pictorial representations of fractions or asked students to draw fractions themselves. Peck and Jencks (1981) asked students to show their thinking with graph paper and paper strips for multiplication and their own drawings for fractions. Pothier and Sawada (1983) asked students to set sticks down across circles to show how they would cut a cake for a designated number of people. Mack's (1990) instructors encouraged students to use fraction strips and circles as long as the students thought them to be necessary. Larson (1980) included number lines in her assessment.

No concrete materials or pictorial representations of fractions were used in studies by Bright and Harvey (1984), Cuneo (1987), Novillis, (1976), and Vance (1986). Singer and Resnick (1992) alluded to jars with red and black marbles in their tasks, but students did not have access to these materials.

Duration of instruction. Some investigators described the duration of their intervention. Peck and Jencks (1981) conducted one 45-minute interview per child. Mack (1990) conducted 11 to 13 half-hour lessons per child over a period of six weeks. Singer and Resnick (1992) conducted one

40-minute session in which students attempted proportion tasks and described their solution strategies. Investigators who used a test as their intervention often did not describe the length of testing time (Bright & Harvey, 1984; Cuneo, 1987; Larson, 1980; Novillis, 1976; Vance, 1986). Pothier and Sawada (1983) did not describe the duration of their interview.

Teacher training. No teacher training was mentioned in these studies. Interviewers were used in many of the studies, but no description of their training is provided. Some of the researchers provided the intervention themselves. Mack (1990) conducted her own instruction. Singer and Resnick (1992) modeled a think-aloud procedure for the students.

Research Designs and Measurement Methods

Most of the studies provided results in a descriptive form rather than in quantitative form. Therefore, research designs are not described. Measurement methods vary, depending on the test chosen by the researcher. Three studies used quantitative analysis. Singer and Resnick (1992) tested the significance of differences with a 2-factor (Quantity Type x Information Type) within-subjects analysis of variance. Larson (1980) used a 2x2 repeated measures analysis of variance to analyze data. Test, retest reliability was reported in the study by Novillis (1976).

Maintenance and Generalization Measures

Neither maintenance nor generalization were explicitly described in most of the studies. Peck and Jencks (1981) encouraged students to use a multiplication strategy that would subsequently help the students solve for common denominators when comparing and adding fractions. The activities used in interviews and the intervention by Mack (1990) required students to build on previous knowledge. This could be interpreted as generalization.

Results

The results of the studies reflected a variety of approaches used by students to represent and interpret fractions. Since the studies were not closely related to small-group or large-group class instruction, generalizability is limited. Bright and Harvey (1984) concluded that the students in their study had specific incorrect strategies of maintaining the difference between the numerator and denominator, dividing numerators, and using multiple strategies to find fraction equivalence. The students also changed strategies depending on whether they solved for the numerator or the denominator. Peck and Jencks (1981) concluded that while 19 of the 20 students could successfully show multiplication with materials, only a few of the students could generalize this skill to comparison and addition of fractions. Novillis (1976) concluded that the subconcepts of part-whole congruent parts were more difficult to learn than part-group, noncongruent parts. She also described certain prerequisites of fraction concepts. Limitations of Novillis' (1976) study included poor test, retest reliability on four subtests, an incomplete hierarchy of skills, and a small sample.

Pothier and Sawada (1983) designed a five-part theory from their study to explain the partitioning process. The researchers asserted that the first four parts of the theory were contained in their data and the fifth part of the theory was the logical next step in the theory. The researchers hypothesized that students first learn to partition quantities into two parts and then into parts numbering powers of two (four parts, eight parts, sixteen parts, and so forth). Next, students learn to partition into even numbered parts. To partition into an odd number of parts, students must learn to begin with something other than a median cut.

Vance (1986) found differences in achievement between sixth- and seventh-grade levels. Male students scored higher on both parts of the fraction comparison test. About half of the subjects in each group in Cuneo's (1987) study were found to have poor ability in computation. Poor performance on estimation of fraction size accompanied poor performance on computation tasks. However, poor performance on computation tasks did not guarantee a lack of understanding of fraction size. Cuneo concluded that an understanding of fraction size is only one of the factors that determine fraction computational skill.

Mack (1990) concluded that the students built on informal knowledge when they could link symbols with real-world situations, but their ability to link fractions and symbols was initially weak. Also, the students' ability to use rote procedures (basic operations such as addition and subtraction) often interfered with their ability to solve fraction problems. Transfer of knowledge from one problem solving situation to another was limited. Discrete models posed more problems than continuous models. Singer and Resnick (1992) concluded that students prefer to work with a part-to-part relationship to a part-to-whole relationship in problem solving with ratios.

Larson (1980) concluded that students found it easier to find points on number lines of one unit. Moreover, students could find fractions more easily on number lines when the units were divided into the same number of parts as the denominator of the fraction. For example, students could find one-third more easily when the unit was divided into three segments rather than six segments. This finding indicated trouble with equivalence concepts. Larson also concluded that students are unable to synthesize fraction concepts.

Lack of data, relatively small samples, limitations in analysis, and little provision of maintenance and generalization procedures weaken the strength of these studies of student thinking in approaches to solving fraction problems. However, the descriptive information that is provided informs the researcher concerning the inconsistencies in fraction learning; the information also prompts further research to determine whether instructional interventions can have an affect on learner strategies and achievement.

Studies from the Rational Number Project

The Rational Number Project is a set of investigations funded by the National Science Foundation and conducted in the 1980s (Behr, Wachsmuth, & Post, 1985). The purpose of the project was to assess children's quantitative concepts of rational numbers. Five major strands were considered by the Rational Number Project: order and equivalence, translations between and within modes of representation, the concept of unit, the quantitative notion of rational number, and the ability to apply rational number concepts to problem-solving situations. Teaching experiments were included in the project and are reported here. The teaching experiments were conducted in schools in Minneapolis, Minnesota, and DeKalb, Illinois. Interviews were often conducted to provide information on students' thinking while approaching a fraction task. The researchers often report student/ researcher dialogue as a component of the data analysis. A total of six studies are reviewed in this section and summarized in Table 3.

Description of the Studies

The studies described in this section investigated student strategies in the area of fractions. Two of the studies (Behr & Bright, 1984; Bright, Behr, Post, & Wachsmuth, 1988) examined students' use of the number line. Behr and Bright (1984) studied the effectiveness of individual instruction in the use of a number line. Behr, Wachsmuth, Post, and Lesh (1984) studied how students determine order and equivalence of fractions. Behr, Wachsmuth, and Post (1985) studied students' strategies for finding fractions close to, but not equal to one. Behr, Wachsmuth, and Post (1988) conducted a teaching experiment where instruction emphasized the use of manipulative objects. Bright, Behr, Post, and Wachsmuth (1988) studied group instruction using the number line. Cramer, Post, and Behr (1989) studied the effect of amount of guidance (low or high) on students' abilities to restructure rational number concepts.

Subject Characteristics

Subjects in each study were students without identified disabilities in fourth or fifth grade. Behr, Wachsmuth, and Post (1988) included 12 students in their study, 6 in the experimental group and 6 in the control group. Cramer, Post, and Behr (1989) included 22 fourth graders in one class

Table 3

Review of Studies from the Rational Number Project

Author(s)	Subjects	Treatment	Measure
Behr & Bright (1984)	5 fourth graders in northern IL	individual instruction in problems with unreduced fractions on a number line instruction lasted 4 days	interview

Purpose: To examine the ways students represent fractions on number lines.

Results: Students had difficulty with partitioning fractions in number line representations. Symbolic and pictorial information presented problems.

Author(s)	Subjects	Treatment	Measure
Behr, Wachsmuth, Post, & Lesh (1984)	12 fourth graders, 6 each at St. Paul, MN and DeKalb, IL	instructional program of 13 lessons each lesson lasted 3 to 8 days entire program lasted 18 weeks	interviews conducted approximately every 8 days during the instructional program

Purpose: To investigate the impact of the use of manipulative materials on learning of fraction concepts.

Results: Responses from the interviews reflected some similarities and differences in how children determine order and equivalence of fractions. In late interviews, 89% of the children's responses indicated that their strategy was appropriate.

Table 3--continued.

Author(s)	Subjects	Treatment	Measure
Behr, Wachsmuth, & Post (1985)	42 fourth and fifth graders schools in Minneapolis and DeKalb, Ill. Data were reported for all eight students at the DeKalb site and 8 of 34 at the Minneapolis site.	teaching experiments of the Rational Number Project Students were asked to put number cards in boxes that made up a fraction equation.	assessment of tasks

Purpose: To examine how well students would perform on the fraction task and to gain insights into successful cognitive mechanisms.

Results: At midassessment, 56% of the students in DeKalb used strategies that would result in correct answers for discrete tasks; 82% of the students in Minneapolis used correct strategies; 0% of the control group used correct strategies. At the final assessment, 81% of the DeKalb students used correct strategies; 100% of the Minneapolis students used correct strategies; 33% of the control group used correct strategies. When completing tasks that used continuous models, fifth graders in control and experimental groups correctly answered all tasks. The results in fourth grade only slightly favored the experimental groups.

Table 3--continued.

Author(s)	Subjects	Treatment	Measure
Behr, Wachsmuth, & Post (1988)	12 fourth graders (6 treatment and 6 control)	teaching experiment that emphasized manipulatives Only continuous models were used during the first half of the program. Discrete models were used after the mid-assessment. Control groups used regular text.	2 assessment interviews, one given half way through the 18 week program and one at the end

Purpose: To investigate students' use of manipulatives in fraction instruction.

Results: During implementation, students were assigned to one of three levels by performance. Students labeled high achievers demonstrated flexibility in their thinking as they approached rational number concepts. Low achievers did not use fraction equivalence to solve problems.

Table 3--continued.

Author(s)	Subjects	Treatment	Measure
Bright, Behr, Post, & Wachsmuth (1988)	43 students fourth and fifth graders in three groups	two clinical teaching experiments and one large group experiment addressed showing and ordering fractions on a number line clinical experiments lasted for 4 days and 8 days in the 30 week program large group experiment lasted for 8 days of instruction in the 30 week program	CTE 1: fraction test given before and after number line lessons CTE 2: fraction test and a number line test were given twice during the intervention Group Exp.: same as in CTE 2

Purpose: To examine how students represent fractions on number lines and how instruction may influence their strategies for representation.

Results: CTE 1: Scores on five of the six nonindependent subscales of the test either remained constant or increased. Scores decreased on the subscale that addressed reducing fractions. Score for the subscales were not reported.

CTE 2: Students had difficulty with unreduced fractions, but mean scores on the posttest improved from 0.75 to 7.75 (8 was the maximum score).

Table 3--continued.

Group Experiment: Posttest scores were significantly higher than pretest scores. Students were more successful with 0-to-1 number lines than with 0-to-2 number lines. Students were more successful when the fraction was given rather than a representation of the fraction. Items that required some repartitioning of fractions were more difficult than those which did not. One item was misdrawn, which may have affected results.

Author(s)	Subjects	Treatment	Measure
Cramer, Post, & Behr (1989)	22 fourth graders grouped by math ability Students were randomly assigned to two groups, high and low guidance. Low guidance group--6 students with high cognitive restructuring --5 with low High guidance group--5 students with high cognitive restructuring ability--6 with low	four parallel lessons were taught 6 class periods of 40 minutes each	Embedded Figures Test and group form (GEFT) for cognitive restructuring abilities pretest and posttest of rational number concepts test

Purpose: To investigate whether an interaction exists between cognitive restructuring ability and levels of teacher guidance.

Results: Interaction was significant ($p<.025$) for continuous perceptual distracter items (which provide information that is not helpful) but not for discrete perceptual distracter items. Students with high cognitive restructuring ability did better with little teacher guidance. Students with low restructuring ability did better with more teacher guidance.

from a predominantly white, middle-to-upper-income district in the Midwest. Behr, Wachsmuth, Post, and Lesh (1984) studied the responses of 12 fourth graders, 6 at each of the experimental sites. Bright, Behr, Post, and Wachsmuth (1988) included fourth and fifth graders in their study. Behr and Bright (1984) studied the responses of 5 fourth graders. Behr, Wachsmuth, and Post (1985) included 42 fourth and fifth graders; data were reported for 8 students at each of these sites.

Experimental Procedures

Random assignment was used in one of the studies. Other researchers described reasons for their selection of students. Cramer, Post, and Behr (1989) randomly assigned students to two groups, high and low guidance. The low guidance group included six students with high cognitive restructuring ability (as measured by the Embedded Figures Test) and five students with low cognitive restructuring ability. The low guidance group included the same types of students.

Behr, Wachsmuth, and Post (1988) matched five of the six fourth graders in their experimental groups to five children in a comparison group. A similar group of six students from the Minneapolis site were included. High, middle, and low achievers were identified by the researchers. Students labeled high achievers demonstrated flexibility in their thinking as they approached rational number tasks. Those labeled low achievers did not use fraction equivalence to solve problems. Four subjects were selected from each category and two were randomly assigned to experimental and control groups. The researchers also included data from fifth graders who were in the pilot group the year before this study as well as data from a fifth-grade control group.

Behr, Wachsmuth, and Post (1985) selected eight students to reflect a full range of ability from the teaching experiment in DeKalb. The researchers also included a homogeneous group of 34 students of average ability. Data for this study were reported for all children at the DeKalb site and 8 of the 34 students at the Minneapolis site.

Bright, Behr, Post, and Wachsmuth (1988) selected students based on teacher recommendation that the students had appropriate competencies in mathematics. Some of the students in this study had been included in a previous extended teaching experiment that was conducted between 1981 and 1983.

Researchers in two studies (Behr & Bright, 1984; Behr, Wachsmuth, Post, & Lesh, 1984) did not describe their selection of students or method of grouping. In summary, the Rational Number Project has provided research with a variety of groupings of students.

Implementation. Studies that included instruction (Behr & Bright, 1984; Behr, Wachsmuth, & Post, 1988; Behr, Wachsmuth, Post, & Lesh, 1984; Bright, Behr, Post, &

Wachsmuth, 1988; Cramer, Post, & Behr, 1989) incorporated instruction that assisted students in problem solving of computation processes. The activities used in the studies were usually not described, with the exception of Cramer, Post, and Behr (1989), who outlined their high guidance and low guidance instruction. Behr, Wachsmuth, and Post (1985) assessed students without instructing them.

One study (Cramer, Post, & Behr, 1989) included a second intervention for its high guidance group where instruction was teacher-centered and direct. Four parallel lessons were presented to the high and low guidance groups. Each lesson consisted of a development phase and a practice phase. The teacher's role in the study determined whether the lesson was high guidance or low guidance. The high guidance lessons allowed little student choice. Large-group instruction was delivered with the teacher presenting the lesson at the overhead projector. The teacher asked questions to help students identify restructuring methods for problems. Immediate feedback was given on each problem. In the low guidance lessons, the teacher provided an introduction and then let students solve problems on their own. The materials included leading questions to encourage student involvement and discovery of common characteristics of incomplete, extraneous, and perceptual distracter problems. Perceptual distractor problems included information that could easily mislead the learner. The teacher did not monitor individual steps in problem solving. At the end of the development phase of the lesson, the teacher led a discussion where students shared their conclusions.

A teaching experiment that emphasized manipulative objects was conducted by Behr, Wachsmuth, and Post (1988). Only continuous models were used during the first half of the experiment. Discrete models were used after the mid-assessment. Control groups used the regular textbook curriculum.

Behr, Wachsmuth, and Post (1985) required students to put number cards in boxes that represented a fraction equation. The objective was to arrange the cards so that the sum was close to, but not equal to 1. The task was presented in two versions. Version 1 of the task used the numbers 1, 3, 4, 5, 6, and 7. Version 2 used the numbers 11, 3, 4, 5, 6, and 7. Version 1 was presented after 20 weeks of instruction. After 27 weeks of instruction, versions 1 and 2 were presented. Students were interviewed regarding their thinking strategies as they completed the tasks.

Bright, Behr, Post, and Wachsmuth (1988) conducted two clinical teaching experiments and one large group experiment on the representation and ordering of fractions on a number line. Students completed number line activities. The researchers used skill tests and videotaped interviews to assess students' ability and understanding. Individual instruction that addressed reducing fractions on a number line was provided in the study by Behr and Bright (1984). Interviews were conducted to determine student thinking strategies.

Behr, Wachsmuth, Post, and Lesh (1984) used an instructional program of 13 lessons that included individual and group activities. Instruction addressed five topics from the Rational Number Project: identifying fractions, fraction equivalence, comparing fractions, adding like fractions, and multiplying fractions. Students were interviewed on 11 occasions, about every 8 days during the instructional period. The interviews addressed the major strands of the Rational Number Project. One strand, order and equivalence, was addressed in this study.

Materials. Tests with pictorial representations of fractions were used in studies by Cramer, Post, and Behr (1989) and Bright, Behr, Post, and Wachsmuth (1988). Behr, Wachsmuth, and Post (1985) used cards with numerals on them and a form board with a fraction template for students to use in solving fraction equations. Behr, Wachsmuth, Post, and Lesh (1984) introduced part-whole interpretation of fractions with circular and rectangular pieces of laminated colored construction paper. Number lines, Cuisenaire rods, poker chips, and paper folding were used later in instruction. Materials were not described in the other studies.

Duration of instruction. All of the studies reported duration of the experiment. Cramer, Post, and Behr (1989) provided lessons in six class periods of 40 minutes each. Behr, Wachsmuth, Post, and Lesh (1984) conducted 11 interviews during their 18-week teaching experiment. The lessons were presented in 3 to 8 days; the entire program lasted 18 weeks. The study by Behr, Wachsmuth, and Post (1985) extended for more than 30 weeks and began in the middle of fourth grade.

Behr and Bright (1984) described instruction that was implemented for 2 years. The first clinical teaching experiment conducted by Bright, Behr, Post, and Wachsmuth (1988) extended for 4 days of instruction near the end of the 18-week program. The second clinical teaching experiment and group experiment each took 8 days of instruction in a 30-week instructional program.

Teacher training. Teacher training is not discussed in any of the studies. However, teacher involvement is described in one study. Cramer, Post, and Behr (1989) used classroom teachers to provide instruction. Studies by Behr, Wachsmuth, Post, and Lesh (1984), Behr and Bright (1984), Bright, Behr, Post, and Wachsmuth (1988), and Behr, Wachsmuth, and Post (1985) do not state if the teacher was involved in the intervention.

Research Designs and Measurement Methods

The studies reflect various methods of data analysis. Cramer, Post, and Behr (1989) used multiple regression to measure interactions between cognitive restructuring ability and treatment. Bright, Behr, Post, and Wachsmuth (1988) analyzed their data with an analysis of variance in the group study and reported mean scores on the clinical teaching experiments. Behr and Bright (1984) reported pretest and posttest scores and significance at the .05 level. Behr, Wachsmuth, and Post (1985) recorded percentages of students using correct versus incorrect strategies for solving fraction equations. Behr, Wachsmuth, Post, and Lesh (1984) and Behr, Wachsmuth, and Post (1988) summarized their findings in narrative form without formal analysis.

Maintenance and Generalization Measures

No maintenance or generalization measures are reported. Bright, Behr, Post, and Wachsmuth (1988) refer to transfer of learning between different tests. However, due to the duration of some of the studies (i.e., up to 30 weeks), it could be assumed that generalization and maintenance of skills were components of the intervention.

Results

Cramer, Post, and Behr (1989) concluded that on continuous tasks, students with high restructuring ability did better with low levels of teacher guidance. The inverse was also true--students with low restructuring ability did better with high levels of teacher guidance.

Behr, Wachsmuth, Post, and Lesh (1984) concluded that students sometimes use inappropriate strategies for comparing fractions, and although this may be helped by instruction, some students continue to lack understanding. Whole number knowledge may contribute to misguided efforts in comparing fractions.

Bright, Behr, Post, and Wachsmuth (1988) concluded that students could improve skills in use of the number line and that certain characteristics of the number line, such as limits of 0 to 1, and simplified representations assist learners in using the number line more efficiently. Behr and Bright (1984) concluded that partitioning and unpartitioning, or finding equivalent fractions and reducing fractions on number lines presented difficulties for students.

Behr, Wachsmuth, and Post (1985) concluded that flexibility of thinking and the use of a reference point enable learners to better apply fraction concepts of order and equivalence. Students who do not have this flexibility tended to be low performers on the presented equivalence tasks. Behr, Wachsmuth, and Post (1985) concluded that high achievers in constructing a sum close to but not exactly one used flexible thinking. Conversely, low achievers in the task did not use thinking strategies or applied strategies inappropriately.

In summary, the experiments conducted by the researchers in the Rational Number Project have provided much descriptive information on students' approaches to fractions in experimental teaching conditions. Most of the studies, however, delivered instruction in small groups (eight students or less) and may not generalize well to the large group instruction that is implemented in many classrooms.

The large group experiment by Bright, Behr, Post, and Wachsmuth (1988) did prove to be successful. The researchers stated that instruction incorporated many activities using number lines, but the activities are not explicitly described. It would seem that effective instruction that could be efficiently implemented in large group settings would enable teachers to teach students to mastery while coping with the logistics of classrooms with 25 or more students.

Studies Addressing the Concrete to Representational to Abstract Instructional Sequence in Mathematics

Jean Piaget (1960) hypothesized that children pass through developmental stages of learning. Other theorists believe that instruction is as much a function of instruction as a function of learner development. The concrete-to-representational (semiconcrete)-to-abstract instructional sequence can be related to early learning theories of Jean Piaget (1960), Lev Vygotsky (1978), Jerome Bruner (1966), Zoltan Dienes (1964), and Robert Gagne (1977). Robert Underhill (1981) described the concrete-to-abstract sequence of instruction in relation to mathematics learning. In this section, learning theories that influence the development of the concrete-to-abstract sequence of instruction are reviewed and implications for fraction instruction are discussed.

Theory of Intellectual Development--Jean Piaget

Piaget described four stages of learning through which every child develops: (a) sensorimotor (birth to 1-1/2 years), (b) preoperational (1-1/2 years to about 7 years), (c) concrete operational (7 years old to adolescence), and (d) formal operational (adolescence through adulthood). Children do not pass through each stage at exactly the same rate; however, Piaget stated that they do pass through the stages in the same sequence.

Sensorimotor stage. The sensorimotor stage reflects a time of great physical and mental growth in the learner. Children at the beginning of the stage exhibit only reflex actions; at the end of the stage, children are in school and are learning basic concepts of space, time, and causality (Souviney, 1989).

Preoperational stage. Symbolic thinking begins at the preoperational stage, though children are not yet able to manipulate symbols mentally. Language takes on richer meaning, and children begin to generalize meaning for certain terms, such as "toys" or "books." Consistency with one-to-one correspondence is difficult to maintain. Children in the preoperational stage cannot conserve length, number, weight, or volume (Souviney, 1989)

Concrete operational stage. Increased ability to demonstrate mental actions characterizes the concrete operational stage as children become able to conserve matter.

Classification and seriation skills become stronger, and effective use of manipulative objects becomes important (Souviney, 1989).

Formal operational stage. Formal reasoning and judgment begin in the formal operational stage of development. Children begin to form hypotheses about learning situations and generalize information to other settings. Symbolic reasoning becomes fully developed in this stage (Souviney, 1989).

Piaget's developmental steps prompt questions regarding the reliance on symbolic concepts inherent in much of fraction instruction (Baker et al., 1994). The appropriateness of teaching fractions in early grades before concrete thought is fully developed could also be debated. An awareness of students' developmental level should prompt teachers and researchers to provide adequate amounts of concrete practice when teaching math concepts to young learners.

Principles of Zone of Proximal Development and Dynamic Support--Lev Vygotsky

Vygotsky (1978) described a range of appropriately challenging learning activities as that child's *range of proximal development*. Within this range, skills are included that the student has already mastered and those skills that require intense direct instruction or expert advice. *Dynamic support* refers to the teacher's ability to adjust levels of assistance as learners become more proficient in skills.

The concept of fractions can be intimidating to any learner. Students are apt to improve achievement more efficiently when they are provided with appropriately challenging activities. Fading of assistance as the learner becomes more proficient in skills provides the student with a greater level of independence (Mercer & Mercer, 1993).

Levels of Representation--Jerome Bruner

Bruner (1966) described three levels of representation of concepts: (a) enactive, (b) iconic, and (c) symbolic. Visual and tactile cues in the physical environment are critically important at the enactive level. At the iconic level, graphic representations convey knowledge. The symbolic level is characterized by increased abstraction of concept, which requires greater mental activity. Souviney (1989) offers that in ideal situations students generalize the abstract concepts in multiple contexts.

Bruner's suggested steps for instruction focus attention on the need for considerable amounts of perceptual support (concrete and pictorial) as the learner develops a concept. Fraction instruction, then, should require much practice at the concrete level to teach the numerous prerequisite skills involved.

Multiple Embodiment Theory--Zoltan P. Dienes

Dienes (1964) proposed a multiple embodiment theory which elaborated on Bruner's approach to presentation of new concepts. Dienes placed a high value on enactive presentation and proposed that concrete experiences must be carefully planned in order to be effective.

The theories of Bruner (1966) and Dienes (1964) relate well to Vygotsky's principles of zone of proximal development and dynamic support (Souviney, 1989). Concrete fraction instruction should be presented in a variety of contexts, as the principle of multiple embodiment is applied. The use of varied contexts encourages the learner to generalize information.

Task Analysis--Robert Gagne

Gagne (1977) supported the use of task analysis for each new skill presented. According to Gagne, a learning task needs first to be placed into a learning outcome category. Gagne described five categories: intellectual skill, cognitive strategy, motor skill, information, or attitude. Next, prerequisite skills must be identified. Gagne also stated that prior learning provides support for new learning, and old capabilities become a part of new capabilities.

Mastery of prerequisite skills is critical to success in fraction instruction. For example, proficiency in basic facts of addition, subtraction, multiplication, and division assist in renaming and computing fractions. Fraction equivalence is also a necessary skill in renaming fractions.

Meaning Theory--William Brownell

Brownell (1928) advanced the theory that students achieve more when they understand mathematical principles than when they learn rote procedures (Riedesel, 1980). Brownell (1928) stated that when mathematics learning is believed to be only a memorization of facts, then teaching is reduced to provision of drill.

Experts in mathematics instruction (Baker et al., 1994; Baroody & Hume, 1991; Cawley, Fitzmaurice-Hayes, & Shaw, 1988) have clamored for meaningful instruction in mathematics. The area of fractions requires meaningful instruction for students to master the many concepts and subconcepts which are involved.

Concrete, Semiconcrete, and Abstract Learning

Underhill (1981) described concrete, semiconcrete, and abstract learning experiences. These experiences relate to the modes of representation described by Bruner (1966).

Concrete experience. Underhill (1981) described concrete learning experiences as "a hands-on experience in which the learner manipulates objects in the presence of the related symbolism; the learner develops meaningful comprehension of the symbols as they relate to his manipulations" (p. 10). Underhill explained that the learner uses manipulative objects in concrete experience to know mathematics enactively, iconically, and symbolically. For example, the student may put three blocks and two blocks together to get five blocks (enactive knowledge). Later, the student may use visual imagery to "see" three blocks and two blocks making five blocks. Finally, symbols are made meaningful by the use of concrete objects.

Semiconcrete experience. Semiconcrete learning experiences are described as visual reinforcement of concepts through visual representations paired with symbols. Associations are built through seeing concrete objects or pictures of objects. Semiconcrete experience helps the student know mathematics iconically and symbolically. Enactive knowledge is precluded because the students do not manipulate objects.

Abstract experience. Abstract learning experiences help the students know mathematics only on the symbolic level because only symbols are involved. Underhill (1981) stated that concrete and semiconcrete experiences will be prerequisites to abstract learning.

Studies Investigating Concrete Instruction in the Development of Math Skills

Two studies were located which investigated the use of concrete models in mathematics instruction. These studies are analyzed and discussed in terms of subject characteristics, experimental procedures, research designs and measurement methods, maintenance and generalization measures, and results in Table 4. Elements which are considered relevant to the present study are analyzed in detail.

Table 4

Review of Studies Investigating Concrete Instruction in the Development of Math Skills

Author(s)	Subjects	Treatment	Measure
Fennema (1972)	95 students 7-8 years old regular ed. assigned to 8 groups	Concrete (Cuisinaire rods) or Symbolic (add on procedure) treatment to teach new skill 14 instructional sessions, same worksheets, problems, and games	One test of recall and two tests of transfer Transfer Test 1 allowed students to use the manipulative objects they had used in their lessons. Transfer Test 2 allowed students to use counters.

Purpose: To investigate the role of concrete and symbolic models.

Results: Both treatment groups did well in recall. The symbolic group performed slightly better than the concrete treatment group. On the Transfer 2 test, all symbolic did better than any concrete. Limitation: Cuisinaire rods were the sole manipulative used in the study.

Table 4--continued.

Author(s)	Subjects	Treatment	Measure
Bebout (1990)	45 students Grade 1 assigned to 1 of 3 levels based on informal strategies	14 lessons, concrete models of word problem first and then open number sentence 3 times per week for 5 weeks	prior to instruction, interviews and math sentence-writing test Posttest: different form of sentence writing test

Purpose: To investigate the theory that children who use concrete models of word problems will be able to symbolically represent word problems with structured number sentences.

Results: Students were successful at representing and solving problems at each level. Students were especially successful in the simplest word problems, where information was presented in the correct order.

Description of Studies

The purpose of each of the two studies was to investigate the effectiveness of the use of concrete models in mathematics instruction. Fennema (1972) and Bebout (1990) used the term symbolic to refer to the abstract level of instruction. Fennema compared the use of concrete instruction to abstract instruction in multiplication. Bebout (1990) investigated whether students who represent word problems with concrete objects would be able to represent word problems symbolically.

Subject Characteristics

The subjects in both studies were elementary students. Fennema (1972) included students ages 7 and 8. Bebout's (1990) subjects were first graders. Subjects in both studies were students in regular education classrooms.

Experimental Procedures

Subjects in each study were selected by whole classroom (Bebout, 1990) or random assignment (Fennema, 1972). Fennema tested students for background knowledge in number recognition, equivalent sets, and addition of one-digit numbers. Students were then randomly assigned to eight groups. Bebout (1990) assigned students to one of three levels of instruction based on their informal strategies in addition and subtraction of word problems: (a) basic (could represent simple addition and subtraction), (b) direct modeling (tried to show the problem), and (c) rerepresentation (tried to ignore the problem structure and solve).

Implementation. Each investigator used 14 lessons to teach the target skill. Fennema (1972) taught four groups with a concrete treatment and four groups with an abstract treatment. Each group in Fennema's study used the same worksheets, problems, and games. Bebout (1990) implemented 14 lessons which used concrete models of word problems and then moved to instruction in open number sentences. The lesson sequence in Bebout's study was review, development, and practice. She described her activities and worksheets in an unpublished manusript. Fennema (1972) described her modeling procedure and provided examples.

Materials. Fennema (1972) used Cuisinaire rods in concrete instruction. Bebout (1990) did not describe the type of concrete objects used in her study.

Duration of instruction. Fennema (1972) did not describe the duration of her study. Bebout's (1990) lessons were conducted 3 times per week for 5 weeks.

Teacher training. Teacher training is not mentioned in either study.

Research Designs and Measurement Methods

Fennema (1972) used an ANOVA to determine significant effects of both types of instruction. Bebout (1990) presented pretest and posttest results by instructional level.

Maintenance and Generalization Measures

Maintenance and generalization were not described by Bebout (1990). Students who were taught in the abstract mode performed higher than students taught in the concrete mode on the transfer tests in Fennema's (1972) study.

Results

Fennema (1972) concluded that both treatment groups could recall information presented. The abstract group performed slightly but not significantly better at the .05 level than the concrete treatment group. On the Transfer II test, all the students in the abstract group did better than any of the students in the concrete group. The author concluded that students had the experiential knowledge through daily living and a curriculum rich in learning via manipulative activities. Bebout (1990) concluded that students were successful at representing and solving problems at each level. Students were especially successful with the simplest word problems, where information was presented in a logical order. Bebout also concluded that the importance of the representational phase is related to efficiency in solving problems and to the representations of problem situations.

Limitations of these two studies include the exclusive use of Cuisinaire rods in Fennema's (1972) study and the possibility of misclassifying students by their informal strategies in Bebout's (1990) study. The assignment of an ability level to an informal strategy quantifies an individual approach to learning. A student may have demonstrated one of a number of approaches in his repertoire. As a result, the student was assessed to be at a certain ability level which might not be a true reflection of his skills.

Studies Investigating the Use of the Concrete, Semiconcrete, and Abstract Sequence in Mathematics Instruction

Studies in the following section measure the effectiveness of the use of the concrete to semiconcrete to abstract sequence of instruction to teach skills in mathematics. Seven studies were located between 1987 and 1993 and are discussed and analyzed in terms of subject characteristics, experimental procedures, research designs and measurement methods, maintenance and generalization measures, and results. Elements of the studies which are considered relevant to the present study are provided in detail.

Table 5

Review of Studies Investigating the Use of the Concrete, Semiconcrete, and Abstract Sequence in Mathematics Instruction

Author(s)	Subjects	Treatment	Measure
Peterson (1987/88)	24 students in elementary and middle school 19 self-contained LD 4 self-contained diagnostic classroom 1 resource	9 scripted lessons in place value 3 concrete 3 representational 3 abstract for treatment group 9 abstract lessons for control group 15 minutes of daily instruction	pre and posttest periodic maintenance checks for 2 weeks after instruction retention test 3 weeks after instruction

Purpose: To investigate the concrete to abstract sequence of instruction of place value in a group setting.

Results: Students in the concrete to abstract group performed significantly better than the control group on acquisition and retention measures.

Table 5--continued.

Peterson, Hudson, Mercer, and McLeod (1990)	Three students in grades 1, 2, and 4 difficulty in math, all identified LD 70% or less on pretest	baseline treatment--CSA sequence was used to teach place value posttreatment (in home schools) maintenance checks were given for two weeks and than discontinued for one week 9-15 days of instruction	pre- and posttest

Purpose: To examine achievement in place value of students with learning disabilities when the concrete to abstract sequence of instruction was used.

Results: Students made gains between pretest and posttest. Retention was successful also. Two of the three subjects demonstrated improvement at the semiconcrete phase.

Table 5--continued.

Author(s)	Subjects	Treatment	Measure
Miller, Mercer, & Dillon (1992)	Fifteen students in grades one through five difficulty in math 10 LD 3 at risk 2 other mild disabilities	CSA sequence applied to addition facts, sums 10-18, subtraction facts, division facts, place value, and coin sums 3 lessons presented at each level daily instruction (20 min.) and four step lesson	pre and posttest at abstract level Short-term retention was measured with a similar posttest one week after the completion of instruction.

Purpose: To validate the CSA sequence and four-step lesson format.

Results: All students achieved mastery of at least 80% and demonstrated mastery on the retention test.

Harris (1992)	13 students with mild disabilities and their normally achieving peers in 6 second grade classrooms	Multiplication Facts 0-81 CSA sequence in teaching math facts, 3 concrete 3 semi 3 abstract and a fluency phase	pre and posttest

Table 5--continued.

Purpose: To examine the effects of using a CSA curriculum in multiplication to teach large classrooms which include students with learning disabilities.

Results: Students with disabilities were able to learn the multiplication and increased fluency in computation.

Miller & Mercer (1993a)	9 students 7-8 years old 5 with LD in math 3 at risk for LD 1 with MMD 6 resource 3 regular ed.	CSA sequence for addition (10-18), division facts with quotients (0-9), and coin sums to 50 cents 3-7 concrete and representational lessons were presented before abstract lessons	3 abstract level probe sheets in addition, division, and coin sums probe given one week after instruction

Purpose: To investigate how much concrete and semiconcrete instruction is needed for students to generalize to abstract numbers.

Results: Four students had more correct than incorrect answers on abstract probes (crossover effect) during the concrete phase of instruction. Five students experienced crossover during semiconcrete instruction.

Table 5--continued.

Author(s)	Subjects	Treatment	Measure
Miller & Mercer, (1993b) I	54 students 9-12 years-old with LD in math 38 self-contained 16 resource mean IQ=95 could add and subtract basic facts; could not multiply 2 single digits	3 concrete lessons 3 representational lessons 10-15 abstract lessons with word problems (lessons were scripted)	pre and posttest with traditional format (Paragraph word problems): 2 without extraneous information, 2 with extraneous information, and 2 student created

Purpose: To investigate the use of the CSA sequence in word problem instruction.

Results: (Means) Overall--84%; Computation--92%; Word problems without extraneous information--97%; Word problems with extraneous information--94%; Student created word problems--60% (gain from pretest mean of 55% on computation and 0% on word problems)

Table 5--continued.

Author(s)	Subjects	Treatment	Measure
Miller & Mercer, (1993b) II	13 students 7-9 years-old with LD in math 10 self-contained 3 resource mean IQ=91 could add 0-9; could not subtract basic facts	21 scripted lessons total 3 concrete 3 representational 3 abstract (all with word problems)	pre and posttest with traditional format (Paragraph word problems): 2 without extraneous information, 2 with extraneous information, and 2 student created

Purpose: To investigate the use of the CSA sequence in word problem instruction.

Results: (Means) Overall--92%; Computation--95%; Word problems without extraneous information--100%; Word problems with extraneous information--86%; Student created word problems 91% (gain from pretest mean of 17% on computation)

Description of Studies

The investigative purpose of each of the studies summarized in Table 5 was to measure the effectiveness of the concrete to semiconcrete to abstract (CSA) sequence of instruction. Peterson (1987/88) compared acquisition of a basic place value skill taught with the CSA sequence to acquisition of the same skill taught exclusively at the abstract level. Peterson, Hudson, Mercer, and McLeod (1990) used the CSA sequence to teach place value and measured effectiveness of the intervention as well as maintenance of the skill. Miller, Mercer, and Dillon (1992) applied the CSA sequence to addition facts, sums 10-18, subtraction facts, division facts, place value, and coin sums. Harris (1992) examined the effects of using a CSA curriculum in multiplication to teach large classrooms which included students with learning disabilities. Miller and Mercer (1993a) also investigated crossover (the point at which students have more correct than incorrect responses on abstract probes) with students instructed with the CSA sequence. Miller and Mercer (1993b) conducted two studies, one with a group of 54 students learning multiplication and one with a group of 13 students learning subtraction. Both studies measured computation and ability to solve word problems.

Subject Characteristics

All of the studies provided subject descriptions that included age, gender, ethnicity, identification of exceptionality, and some statement of ability or achievement. Peterson et al. (1990) included three males in grades 1, 2, and 4 in their study. Each student had been identified as having a learning disability and each experienced difficulty in math. In addition, each student made 70% or less on a pretest in place value. Miller, Mercer, and Dillon (1992) included 15 students with difficulties in math. The IQ range of the students was 63 to 100, and achievement ranged from pre-Kindergarten to grade 3.8.

The multiplication study by Miller and Mercer (1993b) included 54 students with learning disabilities in math. The mean IQ of subjects in this study was 95 with a range of 76-115. The subjects included in the multiplication study could add and subtract basic facts but could not multiply two single digits. Also, the students could not solve word problems. The companion study in subtraction (Miller & Mercer, 1993b) used 13 students who also had learning disabilities in math. The mean IQ for this group was 91. The students in the subtraction study could add digits 0-9 but could not subtract basic facts. Miller and Mercer (1993a) included nine students aged 7 and 8 years. Five of the students had disabilities in math, three were considered at risk for having disabilities in math, and one student had mild mental disabilities.

Harris (1992) included 13 students with disabilities from six second-grade classrooms. The normally achieving peers in each classroom were also included. Peterson (1987/88) included 24 students who ranged in age from 8 to 13 years. All students were identified as having learning disabilities in math.

Experimental Procedures

All of the studies used a pretest to determine that students need to learn the target skill. The pretests were given at the abstract level without materials provided. The pretest in the study by Miller and Mercer (1993b) included computation problems, two word problems without extraneous information, two word problems with extraneous information, and asked students to create two word problems of their own. Miller and Mercer (1993a) provided abstract probes for the skills of addition (0-18), division (0-9), and coin sums to 50 cents. Harris (1992) used multiplication probes of basic facts. Similar forms of the pretests were given as posttests.

Implementation. In each study, systematic, direct instruction in the CSA sequence was provided by the classroom teacher. Three scripted lessons were presented at each phase of instruction, and students were required to reach 80% mastery on independent work to proceed to the next lesson. Miller and Mercer (1993b) expanded the abstract phase of instruction to include 10 to 15 lessons. Miller and Mercer (1993a), Harris (1992), and Peterson (1987/88) included daily 1-minute probes. The study by Peterson et al. (1990) was initiated in a clinical setting, and maintenance checks were continued in each student's home school.

Materials. A variety of materials were used in the studies for the concrete phase of instruction. Teacher-made place value strips and cards were used in the study by Peterson et al. (1990). Materials in other studies included plastic cubes, beans, chips, teddy bear counters, checkers, and other common objects.

Duration of instruction. All of the investigators described duration of instruction. Number of days depended on the students reaching criterion on independent work, which was 80% correct. Peterson et al. (1990) provided treatment for 9 to 15 days. Daily instruction lasted for 20 minutes in the study by Miller, Mercer, and Dillon (1992). Miller and Mercer (1993b) provided 21 scripted lessons in their studies of multiplication and subtraction. Treatment averaged 14 days in the study by Miller and Mercer (1993a). Peterson's (1987/88) treatment phase lasted from 9 to 15 days. Harris (1992) included all of the 21 lessons in Multiplication Facts 0-81 (Mercer & Miller, 1992a) from the Strategic Math Series.

Teacher training. Teachers received inservice training in the concrete to representational to abstract approach in all of the studies. The teachers also received training in how to use the scripted lesson format. Sample dialogue was provided as teachers were encouraged to use the scripts as support for their instruction.

Research Designs and Measurement Methods

A single subject multiple baseline design was used by Peterson et al. (1990), Miller and Mercer (1993a), and Harris (1992). Peterson (1987/88) analyzed data with a 2X3 mixed design with one between (treatment) and one within (performance across time) group factor. A multivariate analysis of variance (MANOVA) was used to find significant differences. Miller, Mercer, and Dillon (1992) and Miller and Mercer (1993b) reported pretest and posttest scores for individuals and mean pretest and posttest scores for the groups.

Maintenance and Generalization Measures

Maintenance and generalization were addressed in each study. Peterson et al. (1990) prompted generalization of skills to a new setting when students returned from the clinical setting to their home school. Periodic maintenance checks continued at the home school for 2 weeks and then discontinued for 1 week. An alternate form of the posttest was given as a retention measure. Miller, Mercer, and Dillon (1992) measured short-term retention with a similar posttest 1 week after completion of instruction. Miller and Mercer (1993b) included create your own problem practice in their study to provide for generalization of problem-solving strategies. Harris (1992) included a fluency phase in her study, and maintenance probes were given after the completion of the study.

Results

Peterson et al. (1990) concluded that the intervention was effective in improving achievement in place value and that students were able to generalize the skill to a new setting. Retention measures also showed increased learning. Miller, Mercer, and Dillon (1992) concluded that the CSA sequence improved performance of individual students in addition facts (sums 10-18), subtraction facts, division facts, place value, and coin sums and that performance on retention measures indicated continued mastery of the skills.

Miller and Mercer (1993b) concluded that computation was significantly improved in both multiplication and subtraction. Although the researchers state that their data were not collected under rigorous experimental conditions, they could conclude that success with word problems was influenced by the following factors: (a) Word problems were introduced at the same time as computation; (b) word problems were related to concrete objects used in lessons; and (c) a match was provided between ability level and problem difficulty.

Miller and Mercer (1993a) concluded that the CSA intervention was effective and that crossover occurred within three to seven concrete and semiconcrete lessons. Harris (1992) concluded that students with disabilities taught multiplication in the classroom group learned with the CSA sequence of instruction and were able to increase their rates of computation. Peterson (1987/88) concluded that the CSA sequence was more effective than abstract instruction only in teaching place value.

Other Studies Which Address the Concrete to Semiconcrete to Abstract Approach to Mathematics Instruction

Harris (1992) reviewed a number of studies in the literature review for her dissertation. Tables and discussion for selected studies have been adapted from her literature review are presented in Tables 6, 7, and 8.

Table 6

Studies that Investigate the CSA Sequence

Author(s)	Subjects	Purpose	Results
Armstrong (1972)	1. 20 ModMD (M.A.=2-4) 2. 67 MMD (M.A.=5.8-11.9)	to compare manipulative vs. non-manipulative in basic number concepts	better learning when representational thought was paired with manipulative
Scott & Neufeld (1976)	145 second graders	to compare concrete and pictorial materials in multiplication	concrete and pictorial equally effective for acquisition of concepts
Prigge (1978)	146 third graders	to compare 2- and 3-dimensional manipulative devices to each other and to a no-manipulative condition in geometric concepts	3-dimensional manipulative condition more effective than other two for low achievers
Smith, Szabo, & Trueblood (1980)	66 first and second graders	to compare manipulative instruction to graphic and abstract instruction in linear measurement	manipulative instruction superior to graphic for acquisition; no difference between manipulative and abstract instruction

Table 6--continued.

Evans (1990)	26 special education and Chapter 1 second and third graders	to compare 2 instructional sequences: manipulative followed by algorithm strategy; algorithm followed by manipulative strategy in subtraction with borrowing	both sequences effective for concepts; concrete first was less efficient

Note. From *The effects of the structured curriculum on the learning of multiplication facts for mainstreamed elementary students with disabilities* (p. 29-97) by C. A. Harris, 1992, University of Florida, Gainesville. Copyright, 1992 by Carolyn Harris. Adapted by permission.

Table 7

Dissertations that Investigate the CSA Sequence

Author(s)	Subjects	Purpose	Results
St. Martin (1975)	99 fifth graders	to compare concrete to abstract vs. semiconcrete to abstract in multiplication and division of fractions	concrete to abstract and semiconcrete to abstract sequences were equally effective for acquisition and retention
Sigda (1983)	52 third graders	to compare concrete to abstract vs. semiconcrete to abstract in multiplication	concrete to abstract was significantly better than semiconcrete to abstract
Sealander (1990)	8 first and second graders ------------- 5 with LD 3 with ED	to investigate skill maintenance and generalization when CSA sequence is discontinued at crossover in subtraction	acquisition, maintenance, and generalization were present when instruction was discontinued at crossover

*adapted from Harris (1992)

Table 8

Master's Projects that Investigate the CSA Sequence

Author(s)	Subjects	Purpose	Results
Kelly (1986)	3 elementary ------------- 2 with LD 1 with MMD	to examine scripted CSA sequence in money recognition to 50 cents	CSA effective for teaching money skills
Robertson (1986)	3 elementary with LD	to examine scripted CSA sequence in addition facts 10-18	CSA effective for acquisition and retention of addition facts
Gayler (1987)	6 elementary Chapter 1	to examine scripted CSA sequence in place value, addition facts 10-18, subtraction facts 0-9	CSA effective for teaching place value, addition, and subtraction
Essinger (1988)	3 elementary ------------- 2 with LD 1 with ED	to examine scripted CSA sequence in subtraction	CSA effective for acquisition, retention, and generalization of subtraction facts
Matson (1988)	3 elementary ------------- 2 with LD 1 regular education	to examine scripted CSA sequence in multiplication facts 0-5	CSA effective for acquisition and retention of multiplication facts

Table 8--continued.

Furtado (1989)	6 elementary ------------- 3 with LD 2 with ED 1 with MMD	to examine scripted CSA sequence in division facts; some students moved to next phase at crossover	Students who moved to next phase at crossover performed as well as students who received full instructional sequence. Students who did not know multiplication required the entire sequence but performed as well as students who knew multiplication.
Natzke (1989)	3 elementary regular education	to examine scripted CSA sequence + mnemonic in subtraction facts (0-9)	CSA + mnemonic effective for teaching subtraction facts
Avino (1990)	4 middle school ------------- 1 with LD 3 with ED	to examine scripted CSA sequence in multiplication word problem solving	CSA effective for acquisition of skills for word problem solution and creation CSA not effective for retention of skills for word problem creation

Table 8--continued.

Clifton (1990)	4 elementary with SED	to examine scripted CSA sequence + mnemonic in multiplication skills	CSA + mnemonic effective for acquisition, retention, and generalization of multiplication skills
Wiers (1990)	3 elementary ------------- 1 with LD 1 with PD 1 at-risk	to examine scripted CSA sequence + mnemonic in subtraction facts 0-9	CSA + mnemonic effective for acquisition, retention, and generalization of subtraction facts

ED	Emotional Disabilities
LD	Learning Disabilities
MMD	Mild Mental Disabilities
PD	Physical Disabilities
SED	Severe Emotional Disabilities

*adapted from Harris (1992)

Harris (1992) asserts that results from the early studies are inconclusive as to whether or not the concrete to semiconcrete to abstract sequence of instruction is effective. More recent studies, however, indicate that the CSA sequence in a controlled format has a positive effect on the learning of many mathematical skills.

Studies Which Use Concrete Instruction or Concrete to Semiconcrete to Abstract Sequence in Fraction Instruction

Two studies were located that applied concrete instruction or the CSA sequence to fraction instruction. The studies are discussed and analyzed in terms of subject characteristics, experimental procedures, research designs and measurement methods, maintenance and generalization measures, and results. Elements which are considered relevant to the present study are analyzed in detail.

Description of Studies

Both studies in Table 9 investigated the effectiveness of concrete instruction in fractions. Harrison, Brindley, and Bye (1989) compared the use of concrete instruction to abstract instruction. Reid (1988) investigated the CSA sequence in fraction instruction in her master's project conducted through the Special Education Department at the University of Florida.

Subject Characteristics

Harrison et al. (1989) included 12-year-old students in their large group study. Reid (1988) included three elementary students aged 8 to 10 in her single subject study. Of the three students, two were identified as having learning disabilities and one was considered low-achieving in mathematics.

Experimental Procedures

Reid (1988) selected students with deficits in fraction concepts. Harrison et al. (1989) assigned comparable pairs of seventh-grade classrooms to experimental and control treatments.

Table 9

Review of Studies Which Use Concrete Instruction or Concrete to Semiconcrete to Abstract Sequence in Fraction Instruction

Author(s)	Subjects	Treatment	Measure
Harrison, Brindley, & Bye (1989)	435 students (12 years old)	experimental-instruction at concrete level, then abstract control-instruction at abstract level	Pretests: Ratio and Proportion Test (Onslow, 1976); Fractional Number Thinking Test (Kieren, 1979) Pre and Posttest: Concepts in Secondary Mathematics (CSMS), Ratio and Proportion Test (Hart, Brown, Kerslake, Kuchemann, & Ruddock, 1984), Attitude Towards Ratios Scale (Brindley, 1980) and (Bell, 1976) Students also completed a Student Questionnaire on Ratios and Student Questionnaire on Fractions (Brindley, 1980).

Table 9--continued.

Purpose: To investigate the effectiveness of a process-oriented teaching approach to fractions and ratios.

Results: Students in the experimental group at the transitional cognitive level performed significantly better than students in the control group. No significant differences were found at the concrete and formal levels. Group x Test interaction favored the experimental group.

Author(s)	Subjects	Purpose	Results
Reid (1988)	3 elementary students 2 with LD 1 low-achieving	CSA sequence for instruction in fractions 3 concrete 3 semi-concrete 3 abstract	pre and posttest retention measures 1 week after instruction and another 2 weeks after instruction

Purpose: To investigate the effect of the CSA sequence in fraction instruction.

Results: The sequence was effective in both acquisition and maintenance of fractions skills.

Implementation. Reid (1988) used systematic instruction with scripted lessons following the CSA sequence. Harrison et al. (1989) used a concrete approach to fractions that led to abstract instruction. In experimental groups, the teacher would discuss the first problem. Small groups would work on similar problems, often with concrete materials. The teacher would question students to help them arrive at the answer rather than giving answers directly. Control classes primarily reflected an examples-rules-exercises cycle, where the teacher would do examples on the board, explain the rules that had been used, give the class similar problems, and work individually with those having trouble. Much time was spent in seatwork from the textbook.

Materials. Both studies used a variety of concrete teaching materials, but only Reid (1988) explicitly described materials used. Those materials included teacher-made and commercial materials. The semiconcrete lessons used worksheets.

Duration of instruction. Instructional treatment time for ratio and fraction units amounted to about 12 weeks in the study by Harrison et al. (1989). A total of nine lessons for each student using the CSA sequence were conducted in the study by Reid (1988).

Teacher training. Reid (1988) delivered instruction in her study. Experimental teachers in the study by Harrison et al. (1989) had inservice during the week before the opening of school. Concrete materials selected by the investigators were introduced, and adaptations for classroom use were discussed and decided upon.

Research Designs and Measurement Methods

Reid (1988) used a multiple baseline single-subject design divided into four stages: baseline, treatment, maintenance, and retention. Harrison et al. (1989) used a three-factor, repeated-measures analysis of variance. The three factors were treatment group (experimental or control), cognitive level (concrete operational, transitional, or formal operational), and test occasion (before or after treatments). _T_ tests measured gains for each cognitive level as assessed by the Ratio and Proportion Test and the Fractional Number Thinking Test. Results from the Attitude Towards Fractions and Attitudes Towards Ratios subscales and the General Mathematics Test were analyzed with a two-factor repeated-measures ANOVA.

Maintenance and Generalization Measures

Reid (1988) included two maintenance probes one week after the end of instruction. Maintenance was not discussed by Harrison et al. (1989).

Results

Reid (1988) concluded that the CSA sequence of instruction was effective in teaching fractions and promoting retention of skills. Harrison et al. (1989) concluded that the concrete approach significantly improved achievement and attitudes toward ratios and fractions. They also asserted that mathematical strategies were enhanced by the process and that computational ability was maintained.

The Use of Games in Mathematics Instruction

Games and play are an important part of a child's daily life and are often incorporated into instruction at school. Randel, Morris, Wetzel, and Whitehill (1992) asserted that most of the information on game effectiveness has been descriptive or based on judgment. These authors also asserted that research on the effectiveness of games instruction has provided mixed results. In this section a summary of studies in the literature review conducted by Randel, Morris, Wetzel, and Whitehill (1992) is provided. Two additional studies on instructional games in mathematics are reviewed.

Randel, Morris, Wetzel, and Whitehill (1992) reviewed the literature on the use of games to teach concepts and skills. The authors examined research summaries on the effectiveness of games that were published between 1966 and 1984. A literature search was conducted to find studies after 1984 and up to the time of their review (1992).

Sixty-eight studies comparing games and simulations to traditional instruction were examined directly or indirectly. The authors found that 38 studies (56%) found no difference in the approaches. Twenty-two studies (32%) favored the use of games/simulations. Five studies (7%) supported games but included questionable factors. Three studies (5%) supported traditional instruction (Randel et al., 1992).

Games were found to be superior to traditional instruction in seven of eight math studies. Specific content and drill and practice seemed to provide good content and skill objectives for games. Although most studies were in the social sciences, the greatest number of the studies showed no difference in student performance. Retention was enhanced more by games/simulations than by traditional instruction, as was student motivation (Randel et al., 1992).

The following studies were not included in the literature review by Randel et al. (1992) (Table 10).

Table 10

Review of Studies Investigating Instructional Games

Author(s)	Subjects	Purpose	Results
Bright, Harvey, and Wheeler (1980)	164 students seventh graders assigned to homogeneous and heterogeneous groups of 3 and 4	probability concept games for half of the students; fractions skill game for other half 20 minute games twice a week for 4 weeks	pre and posttest

Purpose: To study the effects of homogeneous and heterogeneous grouping on instruction through concept and skill games.

Results: Posttest scores were higher than pretest scores for each game-by achievement grouping combination.

Table 10--continued.

Author(s)	Subjects	Treatment	Measure
Bright, Harvey, and Wheeler (1981)	85 students fifth graders in 4 classes 177 seventh graders in 8 classes	game played for 20 minutes twice a week for five weeks to help students order fractions 3 treatments	pre- and posttest

Purpose: To measure the effectiveness of a fraction game in three treatment conditions: (a) game with fraction bars available; (b) game with fraction bars pages available (pictorial representation of fraction bars); and (c) game with no manipulative objects available.

Results: Students improved in all three conditions, but significant differences were found between the groups. Limitations included no control for maturation and a possible Hawthorne effect relating to experience with instructional games.

Table 10--continued.

Author(s)	Subjects	Treatment	Measure
Onslow (1990)	Sample 1 23 students (ages 13 and 14) (UK)	no intervention by teacher in game other than explanation of rules	pre and posttest
	Sample 2 Group 1 18 students (ages 12 and 13)	after game, discussion period	
	Group 2 14 students (ages 13 and 14) mixed ability		

Purpose: To investigate the effects of a game to teach mathematics skills in multiplication and division of decimals.

Results: Posttest scores were significantly better than pretest scores.

Description of Studies

Bright, Harvey, and Wheeler (1980) studied the effects of probability and fraction games on student achievement. A second study by the same authors (1981) compared achievement after students played one of three games: (a) game with fraction bars available, (b) game with fraction bars pages (pictorial representations of fraction bars) available, and (c) game with no manipulative objects available. Onslow (1990) studied the use of games with decimal concepts.

Subject Characteristics

The first study by Bright, Harvey, and Wheeler (1980) included 164 students in seventh grade. The second study (1981) included 85 fifth graders in four classes and 177 seventh graders in eight classes. The students in the second study were described as middle income students. Onslow (1990) studied a group of 23 students in the United Kingdom and a group of 32 students in Canada.

Experimental Procedures

Bright, Harvey, and Wheeler (1980) formed two kinds of groups based on pretest performance. Thirteen combinations were heterogeneous four-person groups; 12 combinations were homogeneous three- and four-person groups. Bright, Harvey, and Wheeler (1981) and Onslow (1990) used class groups.

Implementation. In the study by Bright, Harvey, and Wheeler (1980), students in both homogeneous and heterogeneous groups played probability and fraction games. In their second study (1981), students played a fraction game (*Order Out*) at one of three levels of assistance: concrete, representational, and no assistance. Onslow (1990) designed the game *Shell Shocker* to help students confront the mathematical myth that multiplication always makes things bigger and division always makes things smaller. In the UK study, the teacher explained the game, clarified rules as needed, and then allowed the students to play on their own. After the game, the teacher prompted a discussion about concepts of multiplication and division of decimals included in the game. The teacher summarized information and corrected inaccurate information at the end of the discussion. Students were then given another 15 minutes to play the game. In the Canadian study, the instruction was expanded to two lessons.

Duration of instruction. Students in Onslow's UK study spent approximately 75 minutes on their activities; the Canadian students spent a little more than 100 minutes on their activities. Students in the study by Bright, Harvey, and Wheeler (1980) played games for 20 minutes twice a week for 4 weeks. Students in the second study by Bright, Harvey, and Wheeler (1981) engaged in fraction games for 13 instructional days in 7 consecutive school weeks.

Research Designs and Measurement Methods

Onslow (1990) reported a 2x2 test of significance. Bright, Harvey, and Wheeler (1980) used random assignment to groups. Descriptive statistics and t tests were used to compare pretest-posttest differences. The researchers also ran an analysis of covariance (ANCOVA) to test for interaction. No interaction effects were found. In their second study, Bright, Harvey, and Wheeler (1981) again used a t test to measure pretest-posttest differences. The researchers also used ANCOVAs on the posttest scores to determine significance of treatment, sex, or interaction. The analysis revealed no significant results. However, trends did appear in the analysis. Mean gain scores were higher for girls than for boys for all groups at Grade 5. Only the treatment group that used fraction bars had mean gain scores higher for girls at Grade 7.

Results

Onslow (1990) concluded that students cannot be expected to overcome conceptual obstacles solely by playing a game. Bright et al. (1980, 1981) concluded that games enhanced student achievement in the target skill areas of fractions and probability.

Summary

The research on fraction learning and fraction instruction reveals that, while students may use a variety of approaches to solve fraction problems, they often apply strategies inappropriately. The misapplication of strategies results in a lack of understanding of fractions, a lack of awareness about the uses of fractions, and poor academic performance in the area of fractions and related topics (i.e., decimals). Clearly, research is needed on instructional methods that will foster efficient construction of fraction concepts and provide students with strategies to solve problems that include fractions.

Research on fraction instruction has incorporated use of concrete models with promising results. However, students cannot be expected to use concrete objects indefinitely whenever they are faced with a fraction problem. Research on a sequence of instruction that helps the learner transfer from concrete to abstract concepts is minimal. Further investigation is warranted about instruction that would allow students to solve problems at a higher level of abstraction.

Maintenance of skills over time is crucial to the construction of mathematics concepts that are linked to other concepts. Research on fraction instruction has focused primarily on acquisition of skills. Inability to maintain mastery of skills over time results in increased time needed to acquire new skills, especially by students with learning difficulties. Any research in an area of mathematics as complex and interdependent as fractions should address students' ability to maintain fraction skills and concepts over time.

Research on the concrete to semiconcrete to abstract sequence of instruction indicates that student performance in a variety of mathematics areas is enhanced by explicit sequential instruction, and skills learned are maintained over time. A study of the use of the CSA sequence with fraction equivalence resulted in improved mathematics performance (Reid, 1988). Whereas fraction equivalence is only a part of a unit of study in fractions, the next step in this area of research on the CSA sequence is a study of a broader range of fraction skills.

Moreover, much of the research in both fractions and the CSA sequence has been conducted in small group settings, which allow for increased interaction and individual instruction. Studies with larger groups of students would provide data on the effects of using the CSA sequence to teach fractions in classroom settings.

Typical public school classrooms include 25 to 30 students instructed in large groups for much of the day. As efforts increase to include students with disabilities in general classrooms, those 25 to 30 students may include a significant number of students with mild disabilities or students at risk for school failure. Students with learning problems are most in need of effective instruction in which concepts are clearly developed and opportunities for practice to mastery are provided. Research on effective instructional interventions in large groups that benefit both students with and without disabilities is critical to the advancement of inclusive education efforts.

Research has shown consistently that many students have difficulty learning and maintaining fraction knowledge. National assessments (Lindquist, 1989) demonstrate that many students perform poorly on fraction items. Theory and research indicate that students' ability to use information at various levels of abstraction changes as they learn and grow. The next logical step in this area is to study the acquisition and maintenance of fraction skills in a classroom setting where skills are presented in a concrete to abstract sequence.

CHAPTER III
METHOD

The purpose of Chapter III is to describe the research methodology that was used to investigate three experimental questions about teaching fractions in a large group setting. Specifically, the questions involved comparing the effectiveness of a concrete to semiconcrete to abstract teaching sequence and instruction in the basal text. Initial acquisition, maintenance, and retention of fraction skills was measured. Research methodology, including subject description, hypotheses, instrumentation, materials, procedures, and experimental design and analysis, is discussed.

Subjects

The study was conducted in six fourth-grade classrooms with 125 elementary students. The pool of students included students with disabilities and students identified as gifted as well as students with no identification. There were 56 males and 69 females in the study. The subjects received their math instruction in regular classrooms located in Marion County, Florida. A description of subjects is presented in Chapter IV.

A screening procedure was used to determine students' eligibility to participate in the study. The purpose of the screening was to ensure that students were in need of the skills that were to be taught as recommended by Lessen et al. (1989). In addition, students were screened to determine their level of prerequisite skills in fractions or fraction computation. This screening included a measure of basic multiplication facts; basic division facts; and the use of the symbols to designate greater than (>), less than (<), or equal to (=). To be included in the study, students had to score less than 75% on the fraction screening instrument. Data are reported for students who qualified for inclusion in the study according to the pretest.

Almost 100% of the subjects in the control and treatment groups had little knowledge of the fraction skills assessed by the pretest. One student, however, scored 75% on the pretest. The student participated in the intervention to improve skills, but his data are not included in this study. The mean performance of the control group on the pretest was 17 out of a possible 80 points. The mean performance of the treatment group was also 17 out of a possible 80 points or about 21%. Neither group could successfully compare, add, or subtract fractions at the time of the pretest.

Hypotheses

Several variables were examined in this study to determine possible differences between the concrete to semiconcrete to abstract instruction in fractions and basal text instruction (*Silver Burdett and Ginn Mathematics Level*

4, 1992). The following null hypotheses address experimental questions regarding the two treatment conditions.

H1: There will be no significant difference between students receiving CSA and students receiving basal instruction in scores from a pretest to a posttest administered immediately after treatment (pretest to posttest 1).

H2: There will be no significant difference between students receiving CSA and students receiving basal instruction in scores from a posttest administered immediately after instruction to a posttest administered 1 week following instruction (posttest 1 to posttest 2).

H3: There will be no significant difference between students receiving CSA and students receiving basal instruction in scores on a posttest administered 1 week following instruction to a posttest administered at least 2 weeks following instruction (posttest 2 to posttest 3).

Rejection of the null hypothesis was based on the .05 level of significance.

Instrumentation

In this study, four teacher-made research instruments were used based on the model by Enright (1983). The Enright Diagnostic Inventory of Basic Arithmetic Skills presents mathematical problems in hierarchical form, beginning with simple problems and progressing to more complex. The fraction addition and fraction subtraction subtests were used as references for test construction. Moreover, the fraction chapter tests of *Silver Burdett and Ginn Mathematics*, *Mathematics Today*, *Heath Mathematics*, and *Mathematics in Action* were examined for test item examples.

The instruments were administered to determine skill acquisition immediately after instruction had ended, maintenance of the skills 1 week after instruction had ended, and retention of the skills at least 2 weeks after instruction had ended.

Pretest

The objectives assessed by the pretest were addressed in five sections: (a) identification skills (8 items), (b) comparison skills (24 of 27 items scored for analysis), (c) equivalence skills (16 items), (d) addition skills (16 items), and (e) subtraction skills (16 items) (see Appendix A). Students recorded their answers by drawing circles or lines or filling in blanks as indicated on the test. A Cronbach's alpha was computed from the pretest to provide a reliability measure of .87.

Posttests

The posttests followed the same format as the pretest. Three alternate forms were constructed to measure acquisition and retention of fraction skills immediately following instruction, 1 week after instruction, and as late as possible in the school year (see Appendices B, C, and D). The three alternate forms were randomly assigned by occasion to all subjects for each of the repeated measures to guarantee reliability of the instrument. Each student received each form of the test.

Materials

During the concrete phase of instruction, students in the treatment groups used laminated construction paper circles cut into fraction pieces and strips which represented fractions. Worksheets were also used as part of the lessons. In the semiconcrete phase of instruction, students used fraction reference cards to draw lines representing fractions. Worksheets with fraction representations and/or spaces to draw lines were used in the semiconcrete phase of instruction. Materials used in the abstract phase of instruction included worksheets without fraction representations. All worksheets included space to do problems for demonstration, guided practice, and independent practice problems. Selected worksheets included problems which required the students to generalize their learning to fractions with which they had not yet had concrete or semiconcrete experience. For example, after adding like fractions of halves, thirds, or fourths using fraction pieces, the students might then add tenths or sevenths on their worksheets.

Students in the control groups completed work related to fractions that was assigned by their teacher as part of their textbook instruction in *Silver Burdett and Ginn Mathematics*. This mathematics basal was the textbook adopted by Marion County Schools at the time of the study.

Procedure

The procedure for this study consisted of five phases. Phase one was a pilot study designed to test the intervention in a small group setting. Phase two was a pilot study which tested the intervention in a regular classroom setting. A scripted CSA lesson was used in the intervention (see Appendix E). Phase three consisted of a training inservice for the teachers of the treatment groups. Phase four was the large group study in which the intervention was implemented in four classrooms as treatment and four classrooms as controls. The final phase was posttesting of skill acquisition, maintenance, and retention. An overview of the CSA sequence of instruction will be provided in the next section. Each procedural phase will be described.

Concrete to Semiconcrete to Abstract Sequence of Instructon

The concrete to semiconcrete to abstract sequence of instruction (CSA) incorporates explicit instruction in math skills at each of three levels: concrete (using manipulative objects), semiconcrete (using drawings of objects), and abstract (using no support materials). Each lesson has four distinct parts: (a) advance organizer, (b) demonstration and modeling of the task or skill, (c) guided practice, and (d) independent practice (Mercer & Miller, 1992b).

Advance organizer. The advance organizer allows the learner to access prior knowledge and learning to be able to build new concepts or extend old concepts. The advance organizer also informs the learner of what he will be expected to learn.

Demonstration and modeling of the task or skill. In this part of the lesson, the teacher provides instruction in the target skill by showing students how to perform the task. As the teacher demonstrates, he also tells the students what he is thinking while doing the task (i.e., the teacher uses think alouds). By this method, students hear what they should be thinking when they do the task.

Guided practice. This part of the lesson requires the students to complete the task with monitoring and feedback from the teacher. The students are asked questions about what they should be thinking as they approach the task. As students become successful in completing guided practice problems, the teacher fades her level of monitoring to the point where students are doing most of the problems on their own. This increased independence leads to the final phase, independent practice.

Independent practice. This final part of the lesson requires the students to complete problems autonomously by using what they have learned during the lesson. Data are kept on this part of practice, and results are evaluated to determine whether students have met the criterion for the skill. Criterion is usually set at 80% correct.

Phase One: Pilot Study One

Lessons using the CSA sequence were first taught to three students with learning disabilities at the Multidisciplinary Diagnostic and Training Program (MDTP) at the University of Florida. Both teacher and students responded favorably to the lessons and activities; their feedback assisted the researcher in making revisions.

Phase Two: Pilot Study Two

After testing the CSA intervention with a small group, the lessons were revised. The revised lessons were tested in a large group setting.

Subjects. The students who participated in the pilot study were fourth-grade students at P.K. Yonge Laboratory School in Gainesville, Florida. Two classes participated in the pilot study. One of the classes had previous instruction in fractions during the school year. There were 32 males and 27 females included in the pilot study. White, African-American, and Asian students participated in the pilot study. One student was identified with a learning disability, and one had a hearing impairment.

Design. The purpose of the pilot study was to test the implementation of the intervention in a large group setting.

A pretest-posttest design was used to determine differences in the group between the two tests.

Pretest. Students were pretested in fraction skills before the implementation of the intervention. The pretest consisted of 80 items. Twenty objectives were included on the pretest. (The greater than, less than, and equal to items are considered screening and are not counted in the test scores.) Criterion for mastery was 75% overall. The pretest was administered by the classroom teacher. The two classes differed significantly at the pretest (see Table 11). The class that had received previous instruction in fractions during the school year performed better on the pretest.

Table 11

Groups Means on Pilot Pretest

Group	N	Mean	t	p-value
1 Experience	25	32.04	-6.48	.0001*
2 No Experience	26	16.65		

*Significant at the $p<.05$ level.

Treatment phase. Students were taught to recognize fractions, find equivalent fractions, compare fractions, add fractions, and subtract fractions during the treatment phase. The fraction topics were addressed by instruction using the concrete to semiconcrete to abstract sequence. At least three activities were presented at each level of the instructional sequence. Students who had not mastered the concepts at the end of each phase of instruction were provided with additional practice. Instruction was delivered to the entire class. Instructional time ranged from 30 to 45 minutes per day. Correct and incorrect responses on independent practice were charted on the performance chart developed for the *Multiplication Facts* of the *Strategic Math Series* (Mercer & Miller, 1992a).

Posttreatment phase. After instruction was completed, each student was given a posttest, which was one of three alternate forms of the pretest.

Results. Students in both groups made significant gains in fraction skills on the posttest. As discussed earlier, the two groups were significantly different in pretest scores. At the posttest, however, the two groups were no longer significantly different from each other. Table 12 summarizes the performance of the two groups on the posttest.

Table 12

Groups Means on Pilot Posttest

Group	N	Mean	t	p-value
1 Experience	25	57.72	-1.0669	.2912
2 No Experience	26	52.38		

Teachers of the pilot study groups reported that they preferred the intervention to traditional forms of fraction instruction. Students were motivated by the instructional games. Difficulties were encountered in the transfer of concrete knowledge to worksheet examples. The teachers also made suggestions for the management of materials.

Phase Three: Training Procedures

The three teachers of the experimental groups attended a training inservice before the beginning of the group study in Marion County. At the inservice training, teachers were provided information on effective teaching research, fraction instruction, and the concrete to semiconcrete to abstract teaching sequence. Training in implementation of the instructional activities was also provided. Teachers were assured that the researcher would be on site or available by telephone to provide continuing support and feedback.

Phase Four: Instructional Implementation

A pretest (as described previously) was administered to ascertain the present skill level in fractions of the students. One student scored above mastery, with a score of 80%. After the pretest, instruction began in the control and treatment classrooms. Table 13 summarizes teacher characteristics and instructional conditions for classes in both treatment and control groups.

Control group. Students in the control group were instructed in fraction chapters of their math basal textbook,

Table 13

Teacher Characteristics and Instructional Conditions for Classes in Treatment and Control Groups

	Classes in Treatment Group			Classes in Control Group		
	1T	2T	3T	1C	2C	3C
Teacher characteristics:						
Teacher's gender[a]	F	F	M	F	M	F
Years of teaching experience	19	12	3	27	15	12
Years of teaching experience in fourth grade	9	3	2	7	12	5
Highest degree earned[b]	B	M	B	B	B	B
Instructional conditions:						
Days of fraction instruction	21	17	15	15	15	20
Average length of daily lessons (in minutes)	60	45	60	60	40	60

[a] F = Female; M = Male
[b] M = Master's degree; B = Bachelor's degree

Silver Burdett and Ginn Mathematics Level 4 (1992). Students received between 40 and 60 minutes of instruction daily. Some fluctuations of lesson length occurred depending on class schedules. Instructional days ranged from 15 to 20 days. Instructional materials for the control group subjects included paper folded to represent fractions, fraction bars, and other representations of discreet and continuous fractions. Lessons included demonstration of fraction concepts, and both whole group and independent practice.

Treatment group. Fraction skills were taught to the treatment group using the concrete to semiconcrete to abstract (CSA) sequence and the four-step instructional model (advance organizer, demonstration and modeling, guided practice, and independent practice). Instruction ranged from 45 to 60 minutes. Instruction continued from 15 to 21 days. Criterion for the group was 80% mastery (8 of 10 problems correct) on independent work. Students who did not reach criterion by the end of a phase of lessons (concrete, semiconcrete, abstract) were provided additional practice. Materials for the concrete lessons were laminated construction paper circles cut into fraction pieces. The fraction pieces from the circles were color coded to match designated fractions:

1.	orange	halves
2.	green	thirds
3.	purple	fourths
4.	red	sixths
5.	brown	eighths

Materials for the semiconcrete level lessons were teacher-made worksheets with representations of fractions on them or space to draw representations of fractions. Materials for the abstract level lessons were teacher-made worksheets with no pictures of fractions on them but with ample space for students to draw fraction representations if they chose to do so. A teaching script was used in the lessons (see Appendix E). The number of correct and incorrect responses made during independent practice were recorded by the students on their daily chart.

Control variables. Certain variables were included in the study that helped control for threats to validity. First, students were screened to insure that they were deficient in the skills taught in the study. Teaching scripts were used in the intervention. Schools included in the study were determined to be comparable in SES as reported by the administrative offices of Marion County Schools. Three of the four schools (both treatment and one control) are located within the Ocala city limits. One control school is located near but not in the Ocala city limits. Although the schools differed in size with enrollments of 358 and 714 in treatment schools and 901 and 791 in control schools, classroom sizes were similar. Teachers of the treatment groups were observed to insure the intervention was implemented properly. Teachers of the control groups were observed and interviewed during the study to determine whether they followed the instruction outlined in the basal text.

Phase Five: Posttreatment

After the instruction in the CSA sequence was completed, a posttest to measure skill acquisition was administered to each student in both groups. At the end of 1 week, maintenance was measured with an alternate form of the posttest. Maintenance and retention were measured again at the end of another 2 weeks with another form of the posttest.

Experimental Design and Analysis

The experimental design used in this study is a pretest posttest control group design with one between (treatment) and one within (performance over time) group factor.

A repeated measures analysis of variance (ANOVA) with one between subject factor (treatment versus control) and one within subject factor (performance over time) was computed to determine whether any significant differences were present among the levels of the experimental treatments. A .05 level of confidence was established to determine whether differences were significant and to determine rejection of the null hypotheses. Table 14 depicts the sequence of testing and relates the research hypotheses to the tests administered.

Table 14

Hypotheses Addressed and Measures Given

Groups	Pretest		Posttest 1	Posttest 2	Posttest 3
Treatment	O_0	X	O_1	O_2	O_3
Control	O_0		O_1	O_2	O_3

Question 1--immediate differences are measured by Pretest and Posttest 1 (administered immediately after instruction).

Question 2--maintenance differences are measured by Posttest 1 and Posttest 2 (administered 1 week after instruction).

Question 3--retention differences are measured by Posttest 2 and Posttest 3 (administered as late as possible in the school year).

CHAPTER IV
RESULTS

Introduction

The purpose of this study was to investigate the effects of a concrete to semiconcrete to abstract (CSA) sequence of instruction in fraction concepts and skills. The general question posed in the study was as follows: Does a CSA sequence of instruction affect the acquisition and retention of fraction concepts and skills? To examine this question, the performance of the treatment group, which received a CSA sequence of instruction, was compared to the performance of the control group, which received traditional text-based instruction or instruction that was supplemented by the text. The effects of both types of instruction on fraction concepts and skills of fourth graders were assessed and compared.

This chapter is divided into five sections and presents data acquired in the study that will answer the experimental questions. First, results from measures taken to insure procedural reliability are reported. Second, demographic characteristics of the study participants are described. Third, the univariate analysis of variance results are discussed. Fourth, results of repeated measures ANOVA are reported in relation to the hypotheses. Fifth, related findings including generalization items and social validation are reported.

Reliability of Measurement and Instructional Procedures

Procedures were implemented during the study to insure reliability of measurement and instructional procedures. Teachers of both control and treatment groups were observed to insure procedural reliability. To help standardize test grading, a key was provided to test graders. To help guard against practice and instrumentation effects, students were given a different posttest at each of the three posttesting occasions.

Instructional Procedures

Teachers in the treatment group were observed to determine whether the scripted instruction was being followed. Correct procedure was determined to be the use of the lesson manual, use of appropriate manipulative pieces, and use of appropriate worksheets for guided and independent practice. Figure 1 presents an example of the checklist used for observation. Each teacher in the treatment group was observed by the researcher on at least one occasion during the course of the intervention. During the observation, each teacher followed the directed lesson plan as outlined in the script. Two of the teachers were also observed on two occasions by the learning specialist at their school. As in previous observations, the teachers followed correct instructional procedures.

Checklist for Observation

Date ________ Time ________ Lesson ________

Teacher ______________ Observer __________

Directions: Check each aspect of instruction that applies.

__________ Teacher is using the manual to guide instruction. The script does not have to be read verbatim.

__________ Students are using appropriate manipulative pieces. Fraction pieces or cards are used in concrete and semiconcrete lessons; manipulative pieces are optional in the abstract phase.

__________ Students have appropriate worksheets on hand for guided and independent practice.

Figure 1. Checklist for observation.

Teachers in the control group were observed and interviewed regarding their methods of instruction in fractions. Observations indicated that teachers introduced and discussed fraction concepts within the fourth-grade curriculum. All of the teachers of the control group supplemented textbook instruction with concrete activities. Games were used to reinforce concepts. The teachers reported that they used the textbook primarily as a source for example problems. One teacher reported that she used the textbook to introduce concepts; another reported that the graphics in the textbook were helpful.

Test Grading

Pretests and posttests were graded by students or graduates of the University of Florida, all but one of whom had completed coursework in special education and/or general education. A completed example of the test was given to each scorer to serve as an answer key. Scorers were instructed to give correct answers a 1 and incorrect answers a 0. They also recorded scores in a planned order so that one could go back to individual tests and be able to see achievement on individual items.

Alternate Posttests

Three equivalent forms of the posttest were constructed (see Appendices B, C, & D) and given to students on alternating occasions. The randomization of tests balanced the effect of using different forms of the posttest. For example, in the first posttest given immediately after instruction, one-third of the class received Form A, one-third of the class received Form B, and one-third of the class received Form C. On the second posttest given one week after the end of instruction, those who received Form A during the first test received Form B. Those who had taken Form B first received Form C. Students who first took Form C received Form A. For the final posttest taken as late as possible in the school year, students received the form of the test that they had not seen before. See Appendix F for an example of sequence of tests.

Demographic Characteristics of Participants

A total of six classrooms from four schools participated in the study. The total number of subjects equaled 125. The treatment group totaled 63 students and the control group totaled 62 students. Of the total number of students 69 (55%) were female and 56 (45%) were male. The racial makeup of the total was primarily white, with 104 students (83%). Other racial groups represented were African-American (17 students, 14%), Hispanic (2 students, 1.6%) and Asian (2 students, 1.6%). Lunch status was primarily paid lunch. Ninety-six students (77%) paid for their lunch, 7 students (6%) paid a reduced fee for lunch, and 22 students (18%) had free lunch.

Most of the students (97) had no identified disability, 18 students were identified as gifted, and 11 students were identified with a learning, emotional, speech, or physical disability. Of these 11 students, 1 student had a dual identification of gifted and speech; data from that student are included in the data from students with gifted identification. Table 15 presents a descriptive summary of students in the treatment and control groups.

Table 15

Descriptive Information for Treatment and Control Groups

Descriptors		Group Treatment	 Control
Gender:	male	25	31
	female	38	31
Ethnicity:	white	52	52
	nonwhite	11	10
Lunch status:	free	9	13
	fee-reduced	3	4
	full-priced	51	45
Special education program status:			
	learning disability	3	2
	emotional handicap	1	0
	severe emotional disturbance	0	1
	visual impairment	0	1
	speech/hearing impairment	2	2
	gifted	13	5
	none	45	52

Note. Some subjects participated in more than one special education program. The totals for special education programs do not, therefore, equal the group totals.

Table 16 provides a summary of additional subject characteristics as recommended by Rosenberg et al. (1994) and Lessen, Dudszinski, Karsh, and Van Acker (1989). These

Table 16

Descriptive Information for Subjects with Disabilities

ID#	Age	Ethnicity	Gender	ESE	Lunch Status	IQ			Achievement			
						Verbal	Performance	Full Scale	Reading	Math	Writing	Spelling
T005	10	W	F	SLD	P	100	77	87(R)	99	100	95	101
T018	10	W	F	SLD/CH	P	106	106	106 (III)	84	104	86	89
T026	10	W	M	SLD	P	100	92	96(R)	80 (W)	83 (W)	NA	84 (W)
T064	9	W	M	EH	P	122	117	122(R)	103 (W)	113 (W)	NA	109 (W)
T070	10	W	M	GIF/SCH	P	130	108	122(R)	9 (ST)	8 (ST)		
T323	10	W	M	SCH	P	91	81	85 (III)	88	96	94	93
C107	9	W	M	SLD	P	113	117	117(R)	90	113	100	105
C108	9	W	M	SED	P	122	110	117 (III)	125	108	100	102
C120	11	W	M	SLD/SCH	P	91	120	104(R)	74	92	85	91
C170	10	W	F	SCH	F	87	93	89(R)	4 (ST)	3 (ST)	NA	NA
C211	9	W	F	VIM	F	NA	NA	NA	5 (ST)	5 (ST)	NA	NA

Note. F = free lunch, R = fee-reduced lunch, P = full-priced lunch, ST = stanine, W = Wide Range Achievement Test, R = Wechsler Intelligence Scale--Revised, III = Wechsler Intelligence Scale--Third Edition. SLD = Specific Learning Disability, CH = Chapter 1, EH = Emotional Handicap, GIF = Gifted, SCH = Speech and Hearing, SED = Severe Emotional Disturbance, VIM = Visually Impaired. Data from Student T070 is included in gifted data.

authors propose that age, grade, ethnicity, gender, socioeconomic status (SES), intelligence quotient (IQ) scores, achievement scores, and geographic data be included when describing subjects with learning disabilities. Students are also included in this table if they have a disabling condition other than learning disability.

To determine if there were any preexisting differences between the treatment and control groups, chi-square analyses were conducted to compare the groups by socio-economic status, gender, ethnicity, lunch status, and exceptional student education (ESE) label. For each analysis the .05 level of significance was used.

The control and treatment groups were compared to insure that SES level was equivalent. There is no significant relationship between the treatment groups and SES level ($\underline{X}^2$ = 0.064, df = 1, $\underline{p}$ = 0.800). The same distribution of high SES exists in both groups (i.e., 32% in control and 34% in treatment). Therefore, no sampling bias exists in the selection of groups.

Chi-square ($\underline{X}^2$) tests were used to compare the two groups by gender, ethnicity group, lunch status, and ESE label. The $\underline{X}^2$ revealed no differences between the groups by gender ($\underline{X}^2$ = 1.345, df = 1, $\underline{p}$ = .246). The percentage of females in the control group (50%) is not significantly different from the percentage of females in the treatment group (60%).

Because of the small number of students in each category, all minority populations (African-American, Hispanic, and Asian) were grouped into one category for the ethnicity comparison. The $\underline{X}^2$ revealed no differences between the groups by ethnicity ($\underline{X}^2$ = .040, df = 1, $\underline{p}$ = .842). The percentage of nonwhites in the control group (16%) is not significantly different from the percentage of nonwhites in the treatment group (17%).

Lunch status is used to calculate overall SES for a school. The $\underline{X}^2$ revealed no differences between the groups by lunch status ($\underline{X}^2$ = 1.237, df = 2, $\underline{p}$ = .539). The percentage of students who received free lunch in the control group (21%) is not significantly different from the percentage of students who received free lunch in the treatment group (14%).

The $\underline{X}^2$ did not reveal significant differences between the groups by ESE label ($\underline{X}^2$ = 4.053, df = 2, $\underline{p}$ = .132) at the .05 level. The treatment and control groups are considered to be comparable in numbers of gifted students, students with disabilities, and students without disabilities.

Univariate Analyses

A univariate analysis was performed on pretest scores to determine if there were differences between the groups. No significant differences were found between the treatment and control groups on the pretests ($\underline{t}$ = -3254, df = 123, $\underline{p}$ = .7454). Therefore, the two groups are considered to be comparable in fraction knowledge and skills at the time of the pretest. Table 17 provides the means from the pretest by group.

Table 17

Groups Means on Pretest

Group	N	Mean	t	p-value
Control	62	16.73	-0.33	0.7454
Treatment	63	17.05		

Univariate analyses were also performed on each of the three posttests. Significant differences were found between the treatment and control groups on the first posttest (t = -6.768, df = 103, p = .0001). Significant differences were also found between the treatment and control groups on the second posttest (t = -7.291, df = 93, p = .0001), and on the third posttest (t = -5.895, df = 101, p = .0001). The means of pretest scores for treatment and control groups are summarized in Table 18.

Gifted Students

Although the treatment and control groups did not differ significantly in numbers of students with an ESE label, differences occurred in the percentages of students who were identified as gifted. The control group had 5 gifted students, or 8% of the total group. The treatment group had 13 gifted students, or 21% of the total group. To ascertain

Table 18

Means of Groups by Measure

Measure	Group Means	
	Treatment	Control
Pretest	17.05	16.73
Posttest 1	45.94	28.29
Posttest 2	48.24	27.40
Posttest 3	46.29	28.45

whether the groups would differ without the data from the gifted students, univariate analyses were performed on the data, excluding data from students identified as gifted.

A univariate analysis was performed on pretest scores to determine if there were differences between the groups. No significant differences were found between the treatment and control groups on the pretest ($\underline{t}$ = .461, df = 105, $\underline{p}$ = .6457). Univariate analyses were also performed on each of the three posttests. The number of subjects for each of these analyses was 107. Significant differences were found between the treatment and control groups on the first posttest ($\underline{t}$ = -5.771, df = 76, $\underline{p}$ = .0001). Significant differences were also found between the treatment and control groups on the second posttest ($\underline{t}$ = -5.92, df = 70, $\underline{p}$ = .0001), and on the third posttest ($\underline{t}$ = -4.784, df = 73, $\underline{p}$ = .0001). These results are similar to the results of the groups when data from gifted students are included.

Therefore, the differing percentages of gifted students in the two groups do not present problems to analysis.

Hypotheses

Analysis was conducted to investigate the hypotheses. The repeated measures ANOVA procedure was used to test the three hypotheses. Keppel (1982) states that the repeated measures analysis is the most common design used to study practice, learning, and transfer. The repeated measures ANOVA was selected to examine the three dependent measures and thus reject or fail to reject the three null hypotheses.

Each hypothesis poses a two-fold question. First, did the subjects change significantly in their performance since the previous test? Subjects may experience significant gains, significant losses, or changes that are not significant. Insignificant changes indicate that students are maintaining the knowledge and skills learned during the intervention. In this study, students received instruction before any posttest measures were taken. Therefore, maintenance of skills is an appropriate goal.

The second part of the question addresses the differences in the intervention as reflected in the treatment and control groups. If the test and treatment have a significant interaction, then it is determined that one group behaves differently from the other group. Therefore, follow-up analysis is required to describe the effect of each treatment.

Hypothesis 1

> H1: There will be no significant difference between students receiving CSA and students receiving traditional instruction in scores from a pretest to a posttest administered immediately after treatment (pretest to posttest 1).

The amount of change from pretest to the first posttest is significant for all subjects combined, F(1, 123) = 314.40, p = .0001, that is, performance on posttest 1 is significantly higher than performance on the pretest for all students regardless of treatment group. Table 19 summarizes the repeated measures ANOVA from pretest to posttest 1.

Table 19

Repeated Measures ANOVA for Pretest to Posttest 1

Source	df	F value	p value
TRT	1	30.91	.0001*
Error	123		
Test	1	314.40	.0001*
Test * Trt	1	57.66	.0001*
Error (Test)	123		

*Significant at the $p<.05$ level.

A significant interaction effect occurred between treatment group and test (from pretest to posttest 1), F(1, 123) = 57.66, p = .0001. This indicates that the treatment group performed significantly differently from the control group. Therefore, an analysis was done within each treatment group. The gains of the control group were significant, F(1, 61) = 97.55, p = .0001; the gains of the treatment group were also significant, F(1, 62) = 219.96, p = .0001. The significant interaction and the results of each experimental group indicate that the treatment group performed better than the control group, although both groups improved. Table 20 provides a summary of the results of the repeated measures ANOVA for pretest to posttest 1 by treatment group. Table 21 provides an analysis of the means of each group on posttest 1. Based on these findings, Hypothesis 1 is rejected.

Hypothesis 2

> H2: There will be no significant difference between students receiving CSA and students receiving traditional instruction in scores from a posttest administered immediately after instruction to a posttest administered 1 week following instruction (posttest 1 to posttest 2).

The amount of change from posttest 1 to posttest 2 is not significant for the whole group, F(1, 123) = .94, p = .3345. That is, performance on posttest 2 is not significantly higher (or lower) than performance on posttest

Table 20

Repeated Measures ANOVA for the Pretest to Posttest 1 for Control and Treatment Groups

Source	df	F-value	p-value
Control Group			
Test	1	97.55	.0001*
Error(Test)	61		
Treatment			
Test	1	219.96	.0001*
Error(Test)	62		

*Significant at the $p<.05$ level.

Table 21

Groups Means on Posttest 1

Group	N	Mean	t	p-value
Control	62	28.29	-6.77	.0001*
Treatment	63	45.94		

*Significant at the $p<.05$ level.

1 for the group. Table 22 provides results of the ANOVA from Posttest 1 to Posttest 2.

There is, however, a significant interaction effect between treatment group and test, $F(1, 123) = 4.77$, $p = .0308$. Therefore the relationship between posttest 1 and

Table 22

Repeated Measures ANOVA for Posttest 1 to Posttest 2

Source	df	F value	p value
TRT	1	52.79	.0001*
Error	123		
Test	1	.94	.3345
Test * Trt	1	4.77	.0308*
Error (Test)	123		

*Significant at the p<.05 level.

posttest 2 depends upon treatment group. Table 23 summarizes the repeated measures ANOVA by treatment group. In the time between posttest 1 and posttest 2, the treatment group's performance rose slightly and the control group's performance fell slightly. Performance of the treatment group was not significant, F(1, 62) = 3.33, p = .0728; performance of the control group was also not significant, F(1, 61) = 1.50, p = .2249. The change for neither group is significant, however, the interaction is significant because the change in performance is in different directions.

Table 24 provides an analysis of the means of posttest 2. Hypothesis 2 is not rejected because the changes in each group from posttest 1 to posttest 2 were not significant. Thus, the two groups appear to have maintained the knowledge and skills acquired during instruction.

Table 23

Repeated Measures ANOVA for the Posttest 1 to Posttest 2 for Control and Treatment Groups

Source	df	F-value	p-value
Control Group			
Test	1	1.50	.2249
Error(Test)	61		
Treatment			
Test	1	3.33	.0728
Error(Test)	62		

Table 24

Groups Means on Posttest 2

Group	N	Mean	t	p-value
Control	62	27.40	-7.29	.0001*
Treatment	63	48.24		

*Significant at the $p<.05$ level.

Hypothesis 3

H3: There will be no significant difference between students receiving CSA and students receiving traditional instruction in scores on a posttest administered 1 week following instruction to a posttest administered at least 2 weeks following instruction (posttest 2 to posttest 3).

The amount of change from posttest 2 to posttest 3 is not significant for the whole group, _F_(1, 123) = .46, _p_ = .4975, that is, performance on posttest 3 is not significantly higher (or lower) than performance on posttest 2 for the group. Table 25 provides results of the ANOVA from Posttest 2 to Posttest 3.

Table 25

Repeated Measures ANOVA for Posttest 2 to Posttest 3

Source	_df_	_F_ value	_p_ value
TRT	1	45.08	.0001*
Error	123		
Test	1	.46	.4975
Test * Trt	1	5.10	.0257*
Error (Test)	123		

*Significant at the _p_<.05 level.

There is, however, a significant interaction effect between treatment group and test, _F_(1, 123) = 5.10, _p_ = .0257. Therefore, a follow-up analysis was conducted to determine the performance of each treatment group. Table 26 summarizes the repeated measures ANOVA by treatment group.

Table 26

Repeated Measures ANOVA for Posttest 2 to Posttest 3 for Control and Treatment Groups

Source	df	F-value	p-value
Control Group			
Test	1	1.61	.2090
Error(Test)	61		
Treatment			
Test	1	3.54	.0646
Error(Test)	62		

The follow-up analysis reveals that the relationship between posttest 2 and posttest 3 depends upon treatment group. Changes in the control group are not significant, $F(1, 61) = 1.61$, $p = .2090$; changes in the treatment group are also not significant, $F(1, 62) = 3.54$, $p = .0646$. Students in both the treatment and control groups appear to have maintained their levels of performance when tested at least two weeks after instruction. The interaction is significant because insignificant changes occur in different directions. Table 27 provides an analysis of the means of posttest 3. Based on these findings, Hypothesis 3 is not rejected.

Table 27

Groups Means on Posttest 3

Group	N	Mean	t	p-value
Control	62	28.45	-5.90	.0001*
Treatment	63	46.29		

*Significant at the $p<.05$ level.

Figure 2 provides an illustration of the changes in performance of each group over the four testing occasions. Performance can be described as significant improvement between pretest and posttest 1, with a plateau in scores at posttest 2 and posttest 3. At each posttest, the scores of the treatment group are significantly higher than the scores of the control group.

Related Findings

Further analyses were conducted to answer questions related to the study but not included in the hypotheses. These questions included the following:

1. How did students achieve in terms of ability category, at-risk factors, pretest scores, and gender?
2. Are specific skill areas more or less affected by the intervention?
3. How did mastery of each objective change over time?
4. Did students achieve mastery of the test?

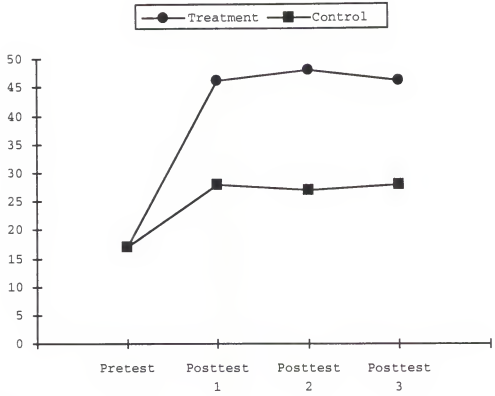

Figure 2. Performance by group across the repeated measures.

5. If students received credit for correct but unsimplified answers, would the results be different?

6. Did students generalize skills to fraction items that they had not seen before?

7. What were students' and teachers' reactions to the treatment?

Achievement for Students by Ability, Pretest Performance, At-risk Factors, and Gender

The data were also analyzed in terms of ability category, at-risk factors, pretest performance, and gender, and are summarized in Table 28.

For each category investigated, the treatment group and control groups did not score significantly different on the pretest. Some groups did perform differently on the posttests. In summary, students who were identified as gifted and students without disabilities seemed to benefit from the intervention. Students with disabilities in the treatment group did not perform better than students with disabilities in the control group. One must also consider the small number of students in the group of students with disabilities and in the group of students identified as gifted. Students were considered to be at risk if they received either free lunch or reduced fee lunch. In this study, at-risk students also seemed to benefit from the intervention. Students were considered to be high pretesters if they achieved at or above the mean of the group, or 17 points. Students were considered to be low pretesters if they achieved below the mean of the group. Both high and low pretesters appeared to benefit from the intervention. Males and females did not perform differently in either group.

Table 28

Achievement by Treatment Group in Ability, At-Risk, Pretest Performance, and Gender Categories

Category	N	Pretest	Posttest 1	Posttest 2	Posttest 3
Total Trt	63	17.05	45.94*	48.24*	46.29*
Total Con	62	16.73	28.29	27.40	28.45
SWD Trt	5	13.80	30.20	29.20	29.80
SWD Con	5	13.40	27.40	29.40	31.00
Gifted Trt	13	22.00	58.38	63.23*	62.00*
Gifted Con	5	23.60	44.00	39.40	45.80
Normal Trt	45	15.98	44.09*	46.02*	43.58*
Normal Con	52	16.39	26.87	26.06	26.54
At-risk Trt	12	15.83	43.42*	47.75*	43.42*
At-risk Con	17	15.53	22.12	22.06	23.29
Hi Pre Trt	35	20.31	52.23*	52.77*	50.51*
Hi Pre Con	33	19.94	31.24	31.70	33.33
Lo Pre Trt	28	12.96	38.07*	42.57*	41.00*
Lo Pre Con	29	13.07	24.93	22.52	22.89
Males Trt	25	18.28	47.76	49.16	46.76
Females Trt	38	16.24	44.74	47.63	45.97
Males Con	31	16.29	29.16	27.42	28.45
Females Con	31	17.16	27.42	27.39	28.45

Note: Means that are significantly higher are indicated with a *. SWD = Students with Disabilities

Skill Areas

Skills on the pretest and posttests were categorized into five areas: identification of fractions, comparison of fractions, fraction equivalence, addition of fractions, and subtraction of fractions. A *t*-test was used to compare scores on posttest 1 by treatment group to determine if there were significant differences in these areas. Significant differences in performance between the two groups occurred in four of the five skill areas: comparison, equivalence, addition, and subtraction. The treatment group who had concrete to semiconcrete to abstract instruction performed significantly better on all skills except fraction identification as measured by posttest 1 than the control group who had traditional instruction. Table 29 provides a summary of the performance of the treatment and control groups in the five skill areas on Posttest 1.

Table 29

Comparison of Skill Area Performance on Posttest 1 by Group

	Treatment Group (n=95)		Control Group (n=88)				
Skill Area	Mean	SD	Mean	SD	*t*	df	*p*
Identification	7.49	1.68	7.81	.44	1.43	71	.1561
Comparison	17.03	4.69	11.71	4.87	-6.22	123	.0001*
Equivalence	8.90	4.68	3.11	3.21	-8.08	110	.0001*
Addition	6.33	5.44	3.24	3.06	-3.92	98	.0002*
Subtraction	6.17	5.06	2.42	2.91	-5.10	99	.0001*

*Significant at the $p<.05$ level.

Mastery of Objectives

Student data were categorized by number of correct responses on each objective, that is, the number and percentage of students who got zero answers correct were calculated as well as the number and percentage of students who got one, two, three, or four answers correct on each objective. Mastery of any objective on this test was considered to be at least three of four answers correct, or 75%. Table 30 provides a summary of mastery performance on each objective for the pretest and three posttest conditions. In general, students in the treatment condition mastered and maintained more objectives than did students in the control group, and did so at an earlier time.

Test Performance

Although students in the treatment group performed significantly better than students in the control group, few students performed well enough to achieve mastery. Table 31 provides a summary of numbers of students who scored at least 75% on the posttests. More students in the treatment group achieved mastery than in the control group.

Unsimplified Responses

When posttesting began, it was suspected that students might do problems correctly up to a certain point but fail to simplify their answers for full credit. To ascertain if students understood fraction concepts and skills but were missing problems because of unsimplified answers, scorers

Table 30

Mastery of Objectives by Treatment and Control Groups Across Testing Conditions

Percentage of Students Who Reached Mastery

Objective	Pretest		Posttest 1		Posttest 2		Posttest 3	
	CON	TRT	CON	TRT	CON	TRT	CON	TRT
Identification								
1	87	84	100	95	98	98	98	94
2	89	84	100	94	97	95	98	97
Comparison								
3	3	10	42	59	34	90	37	57
4	8	10	39	73	44	68	34	76
5	13	5	44	46	45	67	37	63
6	44	46	65	86	66	79	50	79
7	48	33	52	67	61	75	63	78
8	3	3	18	65	13	67	21	63
Equivalence								
9	5	3	44	70	26	71	32	65
10	0	0	3	60	5	54	3	46
11	0	0	11	59	8	65	10	57
12	2	2	8	29	8	44	11	35
Addition								
13	0	2	29	49	29	52	34	52
14	0	2	13	40	6	41	11	37
15	0	0	2	21	3	29	10	29
16	0	0	2	33	5	41	5	40
Subtraction								
17	0	0	2	35	2	35	8	30
18	0	0	24	52	18	46	27	49
19	0	0	2	19	0	19	2	21
20	0	0	2	30	0	29	2	32

Table 31

Total Number of Students Who Achieved Mastery on Posttests by Classroom

Tests	Teachers	
	Treatment	Control
Post 1	16	1
Post 2	22	1
Post 3	22	2

were asked to provide another set of scores that gave credit for correct but unsimplified answers. T-tests were performed on these data, and results were compared to t-tests of data where only correct, simplified answers were counted.

Results of the analyses were similar to the results of the original analyses. Students in neither group performed significantly different when unsimplified correct answers were taken into account.

Generalization Items

Generalization items were included on many of the daily worksheets to allow students to apply their learning to fractions they had not seen before. For example, the worksheet that students used during the lesson on addition of fractions with like denominators included two problems which asked students to first add sevenths and then add tenths. The students' experience with addition of fractions with like denominators up to that point was with halves, thirds, fourths, sixths, and eighths. Performance on the generalization items were mixed. Teachers reported that students were often frustrated with the items and that they felt compelled to help those students with those items.

Social Validation Measures

Social validity of the CSA instruction was measured through questionnaires from students and interviews with teachers. This section includes a summary of the social validation data.

Student Questionnaires

Students in the treatment classes were asked to provide comments and suggestions in an anonymous survey designed by the researcher and distributed by the teacher. Students were asked to describe the things they liked or disliked about the instruction and whether they thought they had learned better, less, or as well as they would if they had used a textbook. Students responded that they liked working with the fraction pieces although they also thought the pieces were hard to keep up with. The students had mixed reactions to the fraction rules and shortcuts. In response to a question that asked students if they had learned more with the fraction intervention than they would have with the math book, one student responded, "Yes, because you get to do it in your hands." A sample of the survey is included in Appendix G.

Teacher Interviews

The researcher contacted the teachers of the treatment groups on a weekly basis through either on-site visits or telephone calls. These contacts allowed teachers to ask questions and give feedback during the study. Moreover, teachers were informally interviewed after the study regarding their perceptions of the intervention and their recommendations for revisions or future implementation. Each of the four teachers reported that they were willing to use the intervention again. The teachers cited the manipulative objects and games as positive aspects of the intervention. Length of lessons was mentioned as a concern.

Summary

The purpose of this study was to determine the effects of the concrete to semiconcrete to abstract sequence of instruction on achievement in fraction skills and concepts. Other areas of interest included achievement in specific skill areas, achievement on specific objectives, performance differences when correct but unsimplified answers were taken into account, generalization items, and students' and teachers' perceptions of the intervention.

Methods were implemented to insure reliability of procedures and scoring. Teachers of the treatment group appropriately used the teacher's manual and materials. Teachers of the control group used the basal textbook either as their primary instructional resource or as a supplement to instruction.

A repeated measures ANOVA was used to test the null hypotheses of no difference between groups from pretest to posttest 1, no difference between groups from posttest 1 to posttest 2, and no difference between groups from posttest 2 to posttest 3. Significant differences between the groups were found between pretest and posttest 1, which resulted in rejection of the first null hypothesis. No significant differences were found between the groups from posttest 1 to posttest 2 which resulted in failure to reject the second null hypotheses. No significant differences were found between the groups from posttest 2 to posttest 3 which again resulted in failure to reject the third null hypotheses.

Related findings were also discussed. Analysis of the treatment and control groups in terms of ability, at-risk factors, pretest performance, and gender revealed some differences between the groups. Analysis of skill areas of the test revealed significant differences between the groups in fraction comparison, equivalence, addition, and subraction. Mastery of specific objectives also differed. The measurement of unsimplified responses proved to be inconclusive, as did generalization measures.

Finally, social validity measures of the CSA intervention yielded positive responses from students and teachers. The teachers and students expressed overall satisfaction with the CSA instruction.

These results suggest that the concrete to semiconcrete to abstract sequence of instruction was more effective for acquisition of fraction concepts and skills than traditional text-based or text-supplemented instruction. The results of retention measures indicate that performance neither increased nor decreased for a time after instruction had ended.

CHAPTER V
DISCUSSION

A discussion of the findings of the present investigation and implications in the study of the effects the concrete to semiconcrete to abstract (CSA) sequence of instruction on fraction concepts and skills is presented in this chapter. The chapter begins with a summary of the hypotheses. A discussion of the theoretical implications of the research findings, limitations of the study, and suggestions for future research are presented.

Summary of the Hypotheses

The general question of the study was as follows: Does instruction in the concrete to semiconcrete to abstract sequence affect the acquisition and retention and use of mathematical concepts and skills in fractions? The following null hypotheses were posited for testing at the .05 level of confidence:

> H1: There will be no significant difference between students receiving CSA and students receiving traditional instruction in scores from a pretest to a posttest administered immediately after treatment (pretest to posttest 1).

A significant difference in the acquisition scores of both the experimental and control groups resulted from pretest to the first posttest. Scores of the treatment group were significantly higher than that of the control group. Thus, the null hypothesis was rejected.

> H2: There will be no significant difference between students receiving CSA and students receiving traditional instruction in scores from a posttest administered immediately after instruction to a posttest administered 1 week following instruction (posttest 1 to posttest 2).

No significant difference was found between posttest 1 and posttest 2 for either the treatment or control group, resulting in a failure to reject the null hypothesis. The treatment group maintained a significantly higher score than the control group after posttest 2.

> H3: There will be no significant difference between students receiving CSA and students receiving traditional instruction in scores on a posttest administered 1 week following instruction to a posttest administered at least 2 weeks following instruction (posttest 2 to posttest 3).

No significant difference was found between posttest 2 and posttest 3 for either the treatment or control group, resulting in a failure to reject the null hypothesis. The treatment group again maintained a significantly higher score than the control group after posttest 3.

Theoretical Implications of the Research Findings

The concrete to semiconcrete to abstract (CSA) sequence of instruction as described by Underhill (1991) provided the theoretical basis for the present study and will be discussed in this section. Other theories and principles which support the CSA sequence also influence the study. These theories include (a) Bruner's levels of representation, (b) Brownell's principle of meaningful instruction, (c) Piaget's theory of cognitive development, (d) Vygotsky's principles of proximal development and dynamic support, (e) Dienes' multiple embodiment principle, and (f) Gagne's task analysis model.

The CSA Sequence, Bruner's Levels of Representation, and Brownell's Meaning Theory

The concrete to semiconcrete to abstract (CSA) sequence of instruction as described by Underhill (1991) was applied in this study through the sequence of activities within the treatment. The activities included three representational levels described by Bruner (1966): (a) enactive, where concrete activities are used to teach; (b) iconic, where graphic representations are used; and (c) symbolic, where information becomes more abstract. These representational levels enhance the meaning of mathematical concepts, which is critical to achievement in mathematics (Brownell, 1928).

Instruction at a concrete level that links student knowledge to new concepts has been recommended in the literature (e.g., Baroody & Hume, 1991; Bezuk, 1988; Engelmann et al. 1991; Nik Pa, 1989). Research on concrete aspects of instruction has included positive results (Armstrong, 1972; Bebout, 1990; Harrison, Brindley, & Bye, 1989; Prigge, 1978; Reid, 1988; Smith, Szabo, & Trueblood, 1980) and inconclusive results (Fennema, 1972). It appears that concrete instruction assists understanding of mathematical concepts, but the effectiveness of the concrete activities are influenced by the delivery of instruction.

The instruction in the treatment group provided students with concrete experience with fraction concepts and skills. Moreover, the instruction taught students strategies to use pictorial representations of fractions and rules and shortcuts to solve fraction problems.

Specific CSA instruction was compared to instruction in fraction concepts and skills that used the school's basal textbook as either a primary resource or as a supplement to instruction. Teachers of students in the control group reported that they used concrete activities to enhance understanding of fraction concepts. The presence of sequential CSA instruction, however, was unique to the intervention.

The data revealed significant gains for both treatment and control groups following instruction. Students who received the CSA instruction, however, made significantly greater gains from pretest to posttest 1 than did the control group. These results are consistent with the findings of previous research (Peterson, 1987/88; Peterson, Hudson, Mercer, & McLeod, 1990; Miller, Mercer, & Dillon, 1992; Harris, 1992; Mercer & Miller, 1992b. Brun, Foy, and Williamson (1994) measured the effects of concrete experience without the CSA sequence on fraction learning of students with disabilities. Results indicated that students did not retain knowledge of fraction skills or concepts when exclusively concrete activities were provided. Thus, the CSA sequence provides support for learners that extend beyond concrete experience.

There were no significant changes in performance for either group at the second posttest and again at the third posttest. This can be viewed as a positive outcome in that students in both groups maintained their level of knowledge. Moreover, the treatment group's achievement was at a significantly higher level than the achievement of the control group. Few students, however, achieved mastery as measured by the posttests. Mastery was considered to be 75%, or at least 60 items correct. Only 26 students in the treatment group and 2 students in the control group achieved mastery on the posttest 3.

Student responses from the treatment group indicate that the games were well received. As recommended by Randel, Morris, Wetzel, and Whitehill (1992) and indicated by the results of this study, the game format was a successful method of providing practice at the enactive level as students traded fractions for equivalent fractions and used fraction pieces to add and subtract fractions of like and unlike denominators.

Piaget's Theory of Cognitive Development

Piaget (1960) described four stages of learning through which every child develops: (a) sensorimotor, (b) preoperational, (c) concrete operational, and (d) formal operational. Children do not pass through each stage at the same rate; however, Piaget purports that children pass through the stages in the same sequence.

The demands inherent in fraction concepts and skills (i.e., multiple operations required, multiple meanings of fractions) brings into question the developmental appropriateness of fractions for students in elementary school. Studies have measured fraction knowledge and skills of students from kindergarten to college level, but there is no consensus at present as to the best time to teach fractions. Mathematics curriculums (e.g., *Silver Burdett & Ginn*, *Mathematics in Action*) address increasingly difficult skills at each grade level, but the skills may not be identical across curriculums.

Experts (Baroody & Hume, 1991; Engelmann et al., 1991) complain that within the spiral curriculum students learn little to a point of mastery and continually lose ground because they do not link knowledge If Piaget's hypotheses are considered, then it is impossible to pinpoint an exact time when learning will take place for every student, especially students with disabilities. Periodic reviews and teaching to mastery are instructional methods that will support cognitive development.

From the results of the pretests of both treatment and control groups, it is unsafe to assume that students will remember concepts taught in previous grades. For example, not all students could identify fractions correctly, a skill that is introduced in kindergarten and is supposedly reviewed each year.

The treatment group in this study addressed fourth-grade fraction objectives that were listed in their mathematics basal textbook. Objectives that had been introduced in earlier grades as well as new objectives were addressed. This component of instruction was included to help students learn skills that they may have missed before.

The skill areas of comparison, equivalence, addition, and subtraction did not reflect achievement to mastery for most of the students. Further research is needed to determine whether this was a result of practice or developmental level.

Another consideration regarding this study should be whether the skills addressed are appropriate for the learners and appropriately sequenced for the learner. In the intervention, students received instruction in addition and subtraction of fractions within each phase: concrete, semiconcrete, and abstract. Students first received instruction in both simple and more complex levels of addition (i.e., addition of like denominators, addition of unlike denominators). Subtraction was addressed after addition. It might be more appropriate to address addition first through the entire CSA sequence and then address subtraction in the same way.

A difficulty encountered in this study was the creation of a means to represent fractions at the iconic level. Fractions could be drawn well enough, but it was difficult to find a pictorial representation that students could easily duplicate. Students' developmental level could also affect their ability to draw fractions.

Vygotsky's Principles of Proximal Development and Dynamic Support

Vygotsky (1978) described a range of activities where students are appropriately challenged, yet not overwhelmed by the difficulty of the task. This range is referred to as the *range of proximal development*. Included in this range are skills that the student has already mastered and skills that require direct instruction or expert advice. Vygotsky also described *dynamic support*, a teacher's adjustment in amount of assistance given to students as the students become more proficient in skills.

The treatment intervention reflects Vygotsky's principles through activities that require both review of concepts and increasingly complex skills. The steps of each lesson from describe and model through guided practice and finally independent practice reflect Vygotsky's dynamic support principle. In this sequence, the teacher is highly involved initially and systematically decreases her leadership as student control increases.

Multiple Embodiment Theory

Dienes (1964) proposed a multiple embodiment theory which places a high value on planning and presentation of enactive activities. Within the present study, enactive activities in the form of games and solving problems on a worksheet are critical components of the instructional program for the treatment group.

Task Analysis

Gagne (1977) supported the use of task analysis for each new skill presented and recognizes that prior learning provides support for new learning. Engelmann, Carnine, and Steely (1991) maintained that students should be taught connections between concepts using the principle of sameness. The use of concrete to semiconcrete to abstract instruction accommodates efforts at task analysis as each phase of instruction is planned to build new skills onto old skills. For example, students in the treatment group first learned to identify fractions, to compare fractions, to find equivalent fractions, and finally to add and subtract fractions. Proficiency in each skill area enhanced abilities in subsequent skill areas, as is reflected in the performance of the treatment group. In each skill area students in the treatment group performed equal to or better than the control group.

In summary, the findings of the present study support the use of a systematic concrete to semiconcrete to abstract sequence of skills and are consistent with learning theories that emphasize concrete experience as a prerequisite to abstract learning. The implications for the population of students with learning problems would become clearer with a larger population to study. This study did not include enough students with disabilities to draw conclusions regarding the effectiveness of the intervention with that particular population.

Limitations to the Present Study

The present study had several limitations. The limitations can be categorized into three broad areas: (a) instructional methodology, (b) population of students with learning disabilities, and (c) grade level of students.

Although the type of instruction for the control groups is considered "traditional," teachers have a certain flexibility at their disposal to teach objectives that they deem important using activities that they choose. Moreover, while a basal series is adopted for use in Marion County, the teachers of the control classes reported that they felt obliged to provide more concrete instruction and relegate the textbook to a secondary role. Some objectives that the treatment group were taught were probably not as fully addressed in the control classroom.

Another limitation of this study is the paucity of students identified with learning or behavior problems that were included in the instruction. Unfortunately, the number of students with learning problems is too small to draw conclusions regarding their achievement. However, a differential effect is indicated in the significant gains made by gifted students and students without disabilities as opposed to the lack of gains by students with disabilities. If students of different ability levels respond differently to an intervention, educators need to consider modifications to instruction that will assist all learners in the classroom.

Revisions of the intervention used in this study will address increased experience at the concrete level and more directed practice at the semiconcrete and abstract levels. For example, some fractions were more easily represented with line drawings than others. Early experience of drawing fractions should be easily achievable, and as students build skills, more complex fractions can be addressed.

A third limitation of this study is the grade level of the subjects. Only fourth graders participated in the instruction. Systematic replication of the procedures is required to generalize results of this study to older or younger students.

Suggestions for Future Research

The findings of this study could contribute to research on long-term effects of the CSA sequence of instruction in fraction concepts and skills. Research is warranted in which the intervention is implemented with students with learning problems. Through such research, appropriate modifications could be determined that would best encourage mastery of skills.

Presentation of an alternative sequence of skills may enhance the performance of students in fraction concepts and skills. Moreover, varying amounts of instructional time in each phase of instruction may influence achievement. Therefore, research is warranted that presents skills in ways that will promote mastery. Considering the differential achievement of subgroups involved in this study, sequence of skills and instructional time needed in each phase may vary depending upon the achievement or ability level of the student, and research in these areas hold implications for instruction in inclusive classrooms.

Teacher support by the researcher was a planned part of this study, as was observations of the lessons. Therefore, a study of the effects of the intervention with less or no involvement by the researcher would be appropriate.

Systematic variations of the groups studied will refine the research on the intervention's effectiveness. In this study, instruction of the control group was left primarily to the discretion of the teacher. Future researchers may investigate more closely the activities of the control group. One way to conduct this research would be to supply control group teachers with the same materials as the treatment group, without the scripted instruction. Another approach would entail providing a script for the control group instruction and activities.

Finally, since fraction concepts and skills are taught throughout the mathematics curriculum, additional research with younger and older students is warranted. Additional research could determine whether a modified program of objectives would help students at other grade levels.

Summary

This study was conducted to examine the effects of a concrete to semiconcrete to abstract (CSA) sequence of instruction on fraction concepts and skills. Results indicated that the CSA sequence was effective in helping students gain skills in fraction identification, comparison, equivalence and computation. In measures of retention at 1 week and at least 2 weeks after the end of instruction, students maintained their level of skills.

The results of this study hold implications for mathematics instruction and mathematics research. Teachers can use a systematic CSA sequence of instruction to teach fraction concepts and skills. Researchers can investigate the CSA sequence to determine how much practice is necessary to develop skills and what type and sequence of objectives provide optimal learning. The refinement of the CSA sequence in fraction instruction should be of interest to general and special educators and researchers.

APPENDIX A
PRETEST

FRACTION PRETEST

Write the name of the fraction that is shaded.

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▲ ▲ ▲ ▲ ▲ △ △

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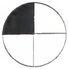

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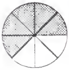

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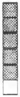

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Circle the fraction that is closest to $\frac{1}{2}$.

$\frac{2}{8}$ $\frac{2}{6}$ $\frac{7}{8}$

$\frac{1}{6}$ $\frac{1}{3}$ $\frac{5}{6}$

$\frac{1}{3}$ $\frac{4}{6}$ $\frac{3}{8}$

$\frac{1}{3}$ $\frac{3}{8}$ $\frac{5}{12}$

Total for Page 1 ________

Circle the fraction that is closest to 0.

$\frac{2}{4} \quad \frac{1}{12} \quad \frac{1}{6}$

$\frac{4}{8} \quad \frac{2}{8} \quad \frac{2}{3}$

$\frac{2}{8} \quad \frac{1}{3} \quad \frac{1}{6}$

$\frac{2}{12} \quad \frac{5}{8} \quad \frac{4}{6}$

Circle the fraction that is closest to 1

$\frac{10}{12} \quad \frac{2}{3} \quad \frac{3}{4}$

$\frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{2}$

$\frac{3}{4} \quad \frac{5}{8} \quad \frac{7}{8}$

$\frac{5}{6} \quad \frac{2}{3} \quad \frac{3}{4}$

Write the correct symbol, >, <, or =.

4 ___ 8

7 ___ 2

3 ___ 3

Compare. Write >, <, or = in the blank.

$\frac{2}{6}$ ______ $\frac{3}{4}$

$\frac{4}{12}$ ______ $\frac{1}{4}$

$\frac{1}{3}$ ______ $\frac{2}{8}$

$\frac{1}{2}$ ______ $\frac{4}{6}$

Compare. Write >, <, or = in the blank.

$2\frac{1}{2}$ _____ $2\frac{1}{3}$

$3\frac{3}{8}$ _____ $2\frac{7}{8}$

$1\frac{1}{2}$ _____ $2\frac{1}{6}$

$3\frac{1}{8}$ _____ $3\frac{2}{6}$

Rewrite the fractions from greatest to least.

			Greatest		Least
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{6}$	_____	_____	_____
$\frac{2}{8}$	$\frac{1}{3}$	$\frac{1}{6}$	_____	_____	_____
$\frac{5}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	_____	_____	_____
$\frac{7}{8}$	$\frac{1}{8}$	$\frac{3}{6}$	_____	_____	_____

Write the equivalent fractions.

$\frac{1}{3}=\frac{}{6}$ $\frac{2}{4}=\frac{}{8}$ $\frac{6}{8}=\frac{}{4}$ $\frac{3}{5}=\frac{}{25}$

Write each mixed number as an improper fraction.

$2\frac{1}{2}=$ _____ $1\frac{5}{8}=$ _____ $2\frac{3}{8}=$ _____ $4\frac{3}{4}=$ _____

Write each improper fraction as a mixed number or whole number.

$\frac{5}{2}=$ _____ $\frac{7}{4}=$ _____ $\frac{11}{8}=$ _____ $\frac{9}{3}=$ _____

Total for Page 3 _______

Write each fraction in lowest terms.

$\frac{7}{14} =$ _____ $\frac{12}{16} =$ _____ $\frac{8}{12} =$ _____ $\frac{7}{21} =$ _____

Add. Write the answer in lowest terms.

$\frac{1}{3}+\frac{2}{3}=$ $\frac{2}{4}+\frac{1}{4}=$ $\frac{4}{8}+\frac{2}{8}=$ $\frac{1}{6}+\frac{2}{6}=$

Add. Write the answer in lowest terms.

$\frac{1}{4}+\frac{2}{8}=$ $\frac{3}{4}+\frac{1}{2}=$ $\frac{1}{3}+\frac{2}{6}=$ $\frac{4}{2}+\frac{2}{8}=$

Add. Write the answer in lowest terms.

$\frac{1}{2}+\frac{2}{3}=$ $\frac{2}{4}+\frac{3}{6}=$ $\frac{2}{3}+\frac{3}{4}=$ $\frac{1}{2}+\frac{1}{3}=$

Add. Write the answer in lowest terms.

$$\begin{array}{r} 2\frac{4}{6} \\ +\frac{3}{2} \\ \hline \end{array} \qquad \begin{array}{r} 4\frac{2}{3} \\ +\frac{2}{6} \\ \hline \end{array} \qquad \begin{array}{r} 2\frac{1}{2} \\ +5\frac{4}{8} \\ \hline \end{array} \qquad \begin{array}{r} 3\frac{1}{2} \\ +1\frac{1}{3} \\ \hline \end{array}$$

Subtract. Write the answer in lowest terms.

$\frac{3}{4}-\frac{1}{4}=$ $\frac{6}{8}-\frac{4}{8}=$ $\frac{11}{12}-\frac{7}{12}=$ $\frac{5}{6}-\frac{2}{6}=$

Subtract. Write the answer in lowest terms.

$$\begin{array}{r}\frac{5}{8}\\ -\frac{1}{4}\\ \hline\end{array}\qquad\begin{array}{r}\frac{3}{6}\\ -\frac{1}{3}\\ \hline\end{array}\qquad\begin{array}{r}\frac{3}{4}\\ -\frac{1}{2}\\ \hline\end{array}\qquad\begin{array}{r}\frac{7}{8}\\ -\frac{1}{4}\\ \hline\end{array}$$

Subtract. Write the answer in lowest terms.

$$\begin{array}{r}\frac{3}{2}\\ -\frac{1}{3}\\ \hline\end{array}\qquad\begin{array}{r}\frac{3}{4}\\ -\frac{1}{3}\\ \hline\end{array}\qquad\begin{array}{r}\frac{4}{6}\\ -\frac{1}{4}\\ \hline\end{array}\qquad\begin{array}{r}\frac{7}{4}\\ -\frac{2}{3}\\ \hline\end{array}$$

Subtract. Write the number in lowest terms.

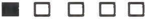
$$\begin{array}{r}2\frac{5}{6}\\ -\frac{1}{3}\\ \hline\end{array}\qquad\begin{array}{r}3\frac{3}{8}\\ -\frac{1}{4}\\ \hline\end{array}\qquad\begin{array}{r}10\frac{5}{8}\\ -7\frac{2}{4}\\ \hline\end{array}\qquad\begin{array}{r}2\frac{1}{2}\\ -1\frac{3}{4}\\ \hline\end{array}$$

Total Test Score ___________

APPENDIX B
POSTTEST 1

FRACTION POSTTEST

Write the name of the fraction that is shaded.



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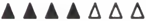

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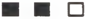

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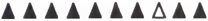

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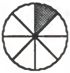

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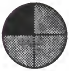

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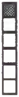

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Circle the fraction that is closest to $\frac{1}{2}$.

$\frac{3}{8}$ $\frac{2}{8}$ $\frac{7}{8}$

$\frac{1}{8}$ $\frac{1}{4}$ $\frac{5}{8}$

$\frac{1}{4}$ $\frac{4}{6}$ $\frac{1}{12}$

$\frac{1}{3}$ $\frac{3}{8}$ $\frac{5}{12}$

Total for Page 1 ________

Circle the fraction that is closest to 0.

$\frac{2}{6}$ $\frac{1}{12}$ $\frac{1}{8}$

$\frac{4}{8}$ $\frac{2}{8}$ $\frac{2}{3}$

$\frac{1}{4}$ $\frac{2}{3}$ $\frac{1}{6}$

$\frac{2}{12}$ $\frac{5}{8}$ $\frac{4}{6}$

Circle the fraction that is closest to 1.

$\frac{5}{12}$ $\frac{7}{8}$ $\frac{3}{4}$

$\frac{2}{3}$ $\frac{3}{4}$ $\frac{1}{2}$

$\frac{6}{8}$ $\frac{3}{6}$ $\frac{2}{8}$

$\frac{7}{8}$ $\frac{2}{6}$ $\frac{3}{4}$

Write the correct symbol, >, <, or =.

6 ___ 9

7 ___ 4

2 ___ 2

Compare. Write >, <, or = in the blank.

$\frac{3}{4}$ ______ $\frac{2}{8}$

$\frac{4}{12}$ ______ $\frac{2}{3}$

$\frac{1}{3}$ ______ $\frac{4}{6}$

$\frac{6}{8}$ ______ $\frac{1}{2}$

Total for Page 2 ________

Compare. Write >, <, or = in the blank.

$5\frac{1}{4}$ ______ $5\frac{1}{3}$ $1\frac{1}{2}$ ______ $2\frac{2}{8}$

$3\frac{3}{8}$ ______ $2\frac{6}{8}$ $3\frac{1}{8}$ ______ $3\frac{1}{2}$

Rewrite the fractions from greatest to least.

			Greatest		Least
$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{6}$	______	______	______
$\frac{5}{8}$	$\frac{1}{2}$	$\frac{2}{6}$	______	______	______
$\frac{2}{8}$	$\frac{1}{3}$	$\frac{1}{6}$	______	______	______
$\frac{7}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	______	______	______

Write the equivalent fractions.

$\frac{1}{4} = \frac{\ }{8}$ $\frac{2}{6} = \frac{1}{\ }$ $\frac{3}{4} = \frac{\ }{8}$ $\frac{3}{5} = \frac{9}{\ }$

Write each mixed number as an improper fraction.

$3\frac{1}{2} =$ ______ $2\frac{3}{8} =$ ______ $1\frac{4}{6} =$ ______ $3\frac{1}{4} =$ ______

Write each improper fraction as a mixed number or whole number.

$\frac{8}{3} =$ ______ $\frac{3}{2} =$ ______ $\frac{9}{8} =$ ______ $\frac{8}{4} =$ ______

Total for Page 3 ________

Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{9}{12}=$ ______ $\frac{8}{16}=$ ______ $\frac{7}{14}=$ ______ $\frac{7}{21}=$ ______

Add. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{1}{3}+\frac{2}{3}=$ $\frac{1}{4}+\frac{1}{4}=$ $\frac{5}{8}+\frac{2}{8}=$ $\frac{1}{6}+\frac{4}{6}=$

Add. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{1}{4}+\frac{3}{8}=$ $\frac{2}{4}+\frac{1}{2}=$ $\frac{2}{3}+\frac{1}{6}=$ $\frac{1}{2}+\frac{2}{8}=$

Add. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{1}{2}+\frac{1}{3}=$ $\frac{1}{4}+\frac{1}{6}=$ $\frac{1}{3}+\frac{1}{4}=$ $\frac{1}{2}+\frac{2}{3}=$

Add. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\begin{array}{r} 1\frac{2}{4} \\ +\frac{1}{2} \\ \hline \end{array}$ $\begin{array}{r} 3\frac{1}{3} \\ +\frac{1}{6} \\ \hline \end{array}$ $\begin{array}{r} 1\frac{1}{2} \\ +3\frac{1}{8} \\ \hline \end{array}$ $\begin{array}{r} 2\frac{1}{2} \\ +\ \frac{1}{3} \\ \hline \end{array}$

Total for Page 4 ______

Subtract. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{5}{4} - \frac{2}{4} =$ $\frac{4}{8} - \frac{1}{8} =$ $\frac{11}{12} - \frac{5}{12} =$ $\frac{5}{12} - \frac{2}{12} =$

Subtract. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{r} \frac{7}{8} \\ -\frac{2}{4} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{6} \\ -\frac{2}{3} \\ \hline \end{array} \qquad \begin{array}{r} \frac{3}{4} \\ -\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{8} \\ -\frac{2}{4} \\ \hline \end{array}$$

Subtract. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{r} \frac{5}{2} \\ -\frac{2}{3} \\ \hline \end{array} \qquad \begin{array}{r} \frac{2}{4} \\ -\frac{1}{3} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{6} \\ -\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} \frac{3}{4} \\ -\frac{2}{3} \\ \hline \end{array}$$

Subtract. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{r} 3\frac{5}{6} \\ -\frac{1}{3} \\ \hline \end{array} \qquad \begin{array}{r} 2\frac{5}{8} \\ -\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 11\frac{7}{8} \\ -7\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 2\frac{1}{2} \\ -\frac{3}{4} \\ \hline \end{array}$$

Total Test Score ________

Total for Page 5 ________

APPENDIX C
POSTTEST 2

FRACTION POSTTEST

Write the name of the fraction that is shaded.

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▲ ▲ ▲ ▲ ▲ △

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■ ■ □ □ □

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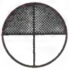

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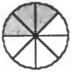

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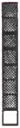

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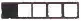

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Circle the fraction that is closest to $\frac{1}{2}$.

$\frac{5}{8}$ $\frac{2}{12}$ $\frac{7}{8}$

$\frac{5}{6}$ $\frac{3}{8}$ $\frac{1}{8}$

$\frac{1}{6}$ $\frac{5}{6}$ $\frac{1}{3}$

$\frac{1}{3}$ $\frac{5}{6}$ $\frac{3}{8}$

Circle the fraction that is closest to 0.

$\frac{1}{6}$ $\frac{1}{12}$ $\frac{1}{4}$

$\frac{3}{6}$ $\frac{1}{12}$ $\frac{1}{8}$

$\frac{2}{4}$ $\frac{1}{3}$ $\frac{4}{6}$

$\frac{2}{8}$ $\frac{2}{3}$ $\frac{3}{8}$

Circle the fraction that is closest to 1

$\frac{2}{12}$ $\frac{1}{8}$ $\frac{3}{4}$

$\frac{2}{12}$ $\frac{5}{8}$ $\frac{1}{4}$

$\frac{7}{8}$ $\frac{2}{6}$ $\frac{5}{8}$

$\frac{2}{8}$ $\frac{3}{6}$ $\frac{3}{4}$

Write the correct symbol, >, <, or =.

1 ___ 3

5___ 2

4 ___ 4

Compare. Write >, <, or = in the blank.

$\frac{2}{4}$ ______ $\frac{1}{8}$

$\frac{1}{6}$ ______ $\frac{2}{3}$

$\frac{2}{3}$ ______ $\frac{2}{6}$

$\frac{1}{8}$ ______ $\frac{6}{12}$

Total for Page 2 ________

Compare. Write >, <, or = in the blank.

$1\frac{1}{2}$ ______ $1\frac{1}{3}$

$1\frac{1}{4}$ ______ $2\frac{1}{6}$

$2\frac{3}{4}$ ______ $3\frac{1}{4}$

$2\frac{2}{8}$ ______ $2\frac{1}{12}$

Rewrite the fractions from greatest to least.

			Greatest		Least
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{2}{6}$	______	______	______
$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	______	______	______
$\frac{4}{8}$	$\frac{1}{6}$	$\frac{1}{12}$	______	______	______
$\frac{3}{8}$	$\frac{1}{12}$	$\frac{1}{2}$	______	______	______

Write the equivalent fractions.

$\frac{1}{4} = \frac{\quad}{12}$ $\frac{1}{3} = \frac{2}{\quad}$ $\frac{2}{4} = \frac{4}{\quad}$ $\frac{1}{5} = \frac{\quad}{20}$

Write each mixed number as an improper fraction.

$1\frac{1}{2} =$ ______ $3\frac{1}{8} =$ ______ $2\frac{2}{6} =$ ______ $2\frac{2}{4} =$ ______

Write each improper fraction as a mixed number or whole number.

$\frac{5}{2} =$ ______ $\frac{5}{4} =$ ______ $\frac{13}{8} =$ ______ $\frac{9}{3} =$ ______

Total for Page 3 ______

Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{6}{12} =$ ______ $\frac{4}{16} =$ ______ $\frac{8}{16} =$ ______ $\frac{4}{24} =$ ______

Add. Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{1}{3} + \frac{2}{3} =$ $\frac{2}{4} + \frac{1}{4} =$ $\frac{3}{8} + \frac{1}{8} =$ $\frac{1}{6} + \frac{2}{6} =$

Add. Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{1}{4} + \frac{3}{8} =$ $\frac{2}{4} + \frac{1}{2} =$ $\frac{2}{3} + \frac{1}{6} =$ $\frac{1}{2} + \frac{2}{8} =$

Add. Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{1}{2} + \frac{1}{3} =$ $\frac{1}{4} + \frac{3}{6} =$ $\frac{1}{3} + \frac{1}{4} =$ $\frac{1}{2} + \frac{2}{3} =$

Add. Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{r} 1\frac{2}{4} \\ +\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} 3\frac{1}{3} \\ +\frac{1}{6} \\ \hline \end{array} \qquad \begin{array}{r} 1\frac{1}{2} \\ +3\frac{1}{8} \\ \hline \end{array} \qquad \begin{array}{r} 2\frac{1}{2} \\ +\ \frac{1}{3} \\ \hline \end{array}$$

Total for Page 4 ________

Subtract. Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{5}{4} - \frac{3}{4} =$ $\frac{7}{8} - \frac{2}{8} =$ $\frac{11}{12} - \frac{5}{12} =$ $\frac{9}{12} - \frac{2}{12} =$

Subtract. Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{r} \frac{7}{8} \\ -\frac{2}{4} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{6} \\ -\frac{2}{3} \\ \hline \end{array} \qquad \begin{array}{r} \frac{3}{4} \\ -\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{8} \\ -\frac{2}{4} \\ \hline \end{array}$$

Subtract. Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{r} \frac{5}{2} \\ -\frac{2}{3} \\ \hline \end{array} \qquad \begin{array}{r} \frac{2}{4} \\ -\frac{1}{3} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{6} \\ -\frac{2}{4} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{4} \\ -\frac{2}{3} \\ \hline \end{array}$$

Subtract. Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{r} 3\frac{5}{6} \\ -\frac{1}{3} \\ \hline \end{array} \qquad \begin{array}{r} 2\frac{5}{8} \\ -\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 11\frac{7}{8} \\ -7\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 2\frac{1}{2} \\ -\frac{3}{4} \\ \hline \end{array}$$

Total Test Score ____________

Total for Page 5 ________

APPENDIX D
POSTTEST 3

FRACTION POSTTEST

Write the name of the fraction that is shaded.

■ ■ □

______

▲ ▲ ▲ ▲ ▲ △

______

■ ■ □ □ □

______

▲ ▲ ▲ ▲ ▲ ▲ ▲ △ ▲

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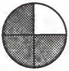

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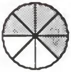

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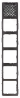

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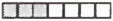

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Circle the fraction that is closest to $\frac{1}{2}$.

$\frac{1}{6}$ $\frac{2}{8}$ $\frac{3}{8}$

$\frac{5}{6}$ $\frac{1}{8}$ $\frac{2}{6}$

$\frac{1}{3}$ $\frac{5}{6}$ $\frac{1}{8}$

$\frac{1}{12}$ $\frac{5}{8}$ $\frac{2}{6}$

Total for Page 1 ______

Circle the fraction that is closest to 0.

$\frac{1}{2}$ $\frac{2}{12}$ $\frac{1}{4}$

$\frac{3}{6}$ $\frac{1}{8}$ $\frac{2}{12}$

$\frac{2}{4}$ $\frac{1}{3}$ $\frac{5}{6}$

$\frac{1}{8}$ $\frac{2}{3}$ $\frac{3}{12}$

Circle the fraction that is closest to 1

$\frac{3}{12}$ $\frac{11}{12}$ $\frac{3}{8}$

$\frac{4}{12}$ $\frac{7}{8}$ $\frac{1}{4}$

$\frac{4}{6}$ $\frac{1}{2}$ $\frac{7}{8}$

$\frac{10}{12}$ $\frac{1}{8}$ $\frac{3}{4}$

Write the correct symbol, >, <, or =.

8 ___ 3

5___ 7

5 ___ 5

Compare. Write >, <, or = in the blank.

$\frac{1}{4}$ ______ $\frac{1}{8}$

$\frac{4}{8}$ ______ $\frac{1}{3}$

$\frac{4}{6}$ ______ $\frac{4}{3}$

$\frac{1}{12}$ ______ $\frac{3}{4}$

Total for Page 2 ________

Compare. Write >, <, or = in the blank.

$5\frac{1}{2}$ ____ $5\frac{2}{3}$ $4\frac{1}{8}$ ____ $2\frac{1}{6}$

$1\frac{5}{8}$ ____ $2\frac{2}{8}$ $2\frac{1}{6}$ ____ $2\frac{2}{8}$

Rewrite the fractions from greatest to least.

			Greatest		Least
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{6}$	____	____	____
$\frac{3}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	____	____	____
$\frac{4}{6}$	$\frac{1}{8}$	$\frac{1}{3}$	____	____	____
$\frac{3}{8}$	$\frac{9}{12}$	$\frac{1}{6}$	____	____	____

Write the equivalent fractions.

$\frac{2}{4} = \frac{}{2}$ $\frac{2}{3} = \frac{4}{}$ $\frac{4}{8} = \frac{}{4}$ $\frac{4}{5} = \frac{16}{}$

Write each mixed number as an improper fraction.

$4\frac{1}{2} =$ ____ $2\frac{3}{8} =$ ____ $3\frac{3}{6} =$ ____ $3\frac{1}{4} =$ ____

Write each improper fraction as a mixed number or whole number.

$\frac{7}{2} =$ ____ $\frac{9}{4} =$ ____ $\frac{16}{8} =$ ____ $\frac{4}{3} =$ ____

Total for Page 3 ____

Write each fraction in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{2}{12} =$ ______ $\frac{12}{16} =$ ______ $\frac{7}{14} =$ ______ $\frac{5}{20} =$ ______

Add. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{1}{3} + \frac{1}{3} =$ $\frac{3}{4} + \frac{2}{4} =$ $\frac{3}{8} + \frac{4}{8} =$ $\frac{1}{6} + \frac{4}{6} =$

Add. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{2}{4} + \frac{3}{8} =$ $\frac{1}{4} + \frac{1}{2} =$ $\frac{2}{3} + \frac{2}{6} =$ $\frac{1}{2} + \frac{5}{8} =$

Add. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{1}{2} + \frac{1}{3} =$ $\frac{1}{4} + \frac{3}{6} =$ $\frac{1}{3} + \frac{2}{4} =$ $\frac{1}{2} + \frac{4}{3} =$

Add. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\begin{array}{r} 4\frac{4}{6} \\ +\frac{1}{2} \\ \hline \end{array}$ $\begin{array}{r} 1\frac{1}{3} \\ +\frac{2}{6} \\ \hline \end{array}$ $\begin{array}{r} 5\frac{1}{2} \\ +2\frac{5}{8} \\ \hline \end{array}$ $\begin{array}{r} 1\frac{1}{2} \\ +\ \frac{1}{3} \\ \hline \end{array}$

Total for Page 4 ________

Subtract. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$\frac{7}{4} - \frac{2}{4} =$ $\frac{5}{8} - \frac{2}{8} =$ $\frac{10}{12} - \frac{2}{12} =$ $\frac{8}{6} - \frac{3}{6} =$

Subtract. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{rrrr} \frac{9}{8} & \frac{5}{6} & \frac{3}{4} & \frac{5}{8} \\ -\frac{2}{4} & -\frac{1}{3} & -\frac{1}{2} & -\frac{1}{4} \\ \hline \end{array}$$

Subtract. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{rrrr} \frac{5}{2} & \frac{2}{4} & \frac{5}{6} & \frac{3}{4} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{3} \\ \hline \end{array}$$

Subtract. Write the answer in lowest terms. Lowest terms is sometimes called simplest form.

$$\begin{array}{rrrr} 3\frac{4}{6} & 4\frac{5}{8} & 7\frac{3}{8} & 2\frac{1}{2} \\ -1\frac{1}{3} & -1\frac{1}{4} & -2\frac{3}{4} & -1\frac{1}{4} \\ \hline \end{array}$$

Total Test Score __________

Total for Page 5 ________

APPENDIX E
LESSONS

Day 3

Goals

To add fractions using manipulative materials
To identify thirds and sixths
To find equivalent fractions of halves, thirds, fourths, sixths, and eighths

Materials

overhead projector and markers
2 sets of fraction pieces for overhead projector (acetate circles of designated fraction color cut into halves, fourths, eighths) and 10 wholes of clear acetate
2 sets of fraction pieces for students (five wholes and 2/2, 4/4, and 8/8)
Learning Sheet 3, one per student
overhead transparency of Learning Sheet 3

whole number dice to be used with mixed number practice or to replace the numerator on fractions rolled on the fraction dice (For example, if you roll a "7" and a 1/4, you could replace the "1" in 1/4 with a "7" to make "7/4.")

The Instructional Procedures

Give an Advance Organizer

1. Tell the students what they will be doing and why.

Sample dialogue:
Yesterday we added the fractions 1/2, 1/4, and 1/8. We also played a game with these fractions. You did a good job on your game. Today we're going to continue with our game and practice making different kinds of trades. We're going to learn about two new fractions. Adding fractions and finding other names for fractions helps us use them better.

Describe and Model Task or Skill

This section will introduce thirds and sixths and show fraction equivalence between the two types of fractions. The instruction is similar to introduction of the other fractions.

1. Teacher asks a question and teacher answers question to help students identify thirds and sixths. Put whole circles and thirds and sixths fraction pieces on the overhead projector. Explain to the students that you will be asing questions and then answering them yourself. This will help them see how to think when solving problems.

Sample dialogue:
I'm going to look at some fraction pieces. (Place three one-third pieces over the one whole.) Is each of these pieces the same size? Yes, they are the same size. When I put these three pieces on top of the one whole, I cover it all up. How many green pieces will cover the one whole? Three is my answer. I'll write "3" by Problem 1 on my learning sheet. You can write "3" by Problem 1 also. (Monitor to see that students have written the answer.) Each green piece is called one third. When something is divided into three equal pieces, the pieces are each called one-third. What are the pieces called? (Elicit the response, "One-third.") Everybody say, "One-third." Good.

2. **Teacher asks question and students help provide answer to help students identify thirds and sixths.**

Sample dialogue:
Here's another fraction. Let's identify it together. Place six one-sixth pieces over the one whole. (Model this on the overhead projector.) Is each of these pieces the same size? (Elicit the response, "Yes, they are the same size.") When we put these six pieces on top of the one whole, do we cover it all up? (Elicit the response, "Yes.")
How many red pieces will cover the one whole? (Elicit the response, "6.") Right, everybody say, "Six sixths equals one whole." Let's write "6" by Problem 2 on our learning sheets. (Model and monitor to see that students do this.) Each red piece is called one-sixth. When something is divided into six equal pieces, what is each piece called? (Elicit the response, "one-sixth.")

At this point, you will show students how thirds and sixths are related fractions. Students may need to practice the equivalence between sixths and thirds.

3. **Teacher asks question and students answer question to help students understand fraction equivalence.**
Put two sixths and two thirds pieces on the overhead.

Sample dialogue:
Now I see that I have two sixths here and two thirds here. If I was playing a game with this, I could wait to draw one-third, or wait to draw four more sixths, or maybe do something with the pieces I have.
Let's see, I've got two sixths and two-thirds. Let me move those around. (Put pieces together in a circle.) Hey, they make a whole circle if I put them together! I

could make more trades if I could combine pieces, but, our rule is that only the same color pieces can be traded in for ones. What can I do? Wait a minute, I think my two sixths might be the same as one third. Let me check it out. (Put the two sixths on top of the third to show that it does match.) Remind students that they did this in a previous lesson to see how many fraction pieces covered the whole.) If I take my two sixths pieces and trade it in for one third, I'll have all one color! We call this renaming fractions. In this example, I gave my two sixths a new name, one third. I put the two-sixths in its **simplest form**.

4. **Teacher demonstrates the special case of halves and thirds.** In this section, you will show the students that when they add or subtract with halves and thirds, they will need to trade each for sixths. Show one-half and one-third on the overhead.

   Sample dialogue:
   I have one-half and one-third. Put those pieces in your working space. I would like to be able to add these, but I can't trade them for each other. There is a fraction we know about, however that we can trade with each of them. Can anyone guess what that fraction is? (Elicit the response, "Sixths.") You are right! Let's use our fraction pieces to be sure. First, find out how many sixths it takes to cover one-half. (Pause.) How many sixths does one-half equal? (Elicit the response, "Three-sixths.") Right. Everybody say, "Three-sixths equal one-half." Now, find out how many sixths it takes to cover one-third. (Pause.) How many sixths does it take to equal one-third? (Elicit the response, "Two-sixths.") Right. Everybody say, "Two-sixths equal one-third." You will have problems later on where you will add or subtract halves and thirds. You will need to change both fractions to sixths to be able to do the work.

Conduct Guided Practice

In this section, students will play the fraction addition game with all of the fractions that have been introduced. The game should continue for 15-20 minutes. Your role is to monitor the games to see that students are making correct trades. The students will also be monitoring each other.

Describe and Model

1. **Teacher asks a question and teacher answers question to help students add fractions.**

Sample dialogue:
What we have been doing throughout the game is *adding fractions*. Let's do some fraction addition problems now. Look at Problem 3. I'll show you how to do this one, and you can write the answer. The problem says, "1/3 plus 1/3 equals blank." I can do that problem by using my fraction pieces. I'll take a green piece, which is one third, and then I'll take another green piece and see what I have. I have two green pieces, or 2/3 Let's write that fraction on our learning sheet.

2. **Teacher asks a question and students help provide answer to help students add fractions.**

Sample dialogue:
Let's do one together. Look at Problem 4. It says, "1/6 plus 2/3 equals blank." Are the denominators the same? (Elicit the response, "No.") So we need to trade. What do we need to trade? (Elicit the response, "Trade thirds for sixths.") That's right, you need each third for two sixths. How many sixths are equivalent to two-thirds? (Elicit the response, "Four.") Make your trade. (Monitor to see that students have done this.) What do you have altogether? (Elicit the response, "Five-sixths.") Right. Let's write that on our learning sheet for Problem 4. (Monitor.) Before we are finished with the problem, we need to be sure that it is in simplest form. Can five-sixths be put in a simpler form? (Elicit the response, "No.") Everybody say, "Five-sixths is in simplest form."

Conduct Independent Practice
These problems can be scored to help determine mastery of this skill.

Sample dialogue:
Problems 5 through 14 are for you to do on your own. Use your fraction pieces to find the answers. If the problems have two blanks, that is a cue that you need to make additional trades and you will need to think about what you did in the game. Problems 13 and 14 are a little different. Think about them carefully, because you haven't seen these fractions before.When you finish Problem 14, put your pencil down.

What are you going to do?

Repeat directions if necessary.

Circulate and monitor the students' work.

Demonstration and Modeling/Guided Practice

Fractions--relation to one

1. How many green pieces cover one whole? _______

2. How many red pieces cover one whole? _______

3. $\begin{array}{r}\frac{1}{3}\\ +\frac{1}{3}\\ \hline \end{array}$

_____

4. $\begin{array}{r}\frac{1}{6}\\ +\frac{2}{3}\\ \hline \end{array}$

_____

Independent Practice

5. $\begin{array}{r}\frac{1}{3}\\ +\frac{4}{6}\\ \hline \end{array}$

_____=_____

6. $\begin{array}{r}\frac{4}{8}\\ +\ \frac{1}{2}\\ \hline \end{array}$

_____=_____

7. $\begin{array}{r}\frac{2}{6}\\ +\frac{3}{6}\\ \hline \end{array}$

_____=_____

8. $\begin{array}{r}\frac{2}{3}\\ +\frac{7}{6}\\ \hline \end{array}$

_____=_____

9. $\begin{array}{r}\frac{2}{3}\\ +\frac{1}{2}\\ \hline \end{array}$

_____=_____

10. $\begin{array}{r}\frac{5}{6}\\ +\frac{1}{3}\\ \hline \end{array}$

_____=_____

11. $\begin{array}{r}\frac{7}{8}\\ +\frac{2}{4}\\ \hline \end{array}$

_____=_____

12. $\begin{array}{r}2\frac{3}{6}\\ +3\frac{1}{3}\\ \hline \end{array}$

_____=_____

13. $\begin{array}{r} 1\frac{2}{5} \\ +3\frac{3}{5} \\ \hline \end{array}$

_____ = _____

14. $\begin{array}{r} \frac{3}{7} \\ +\ \frac{2}{7} \\ \hline \end{array}$

_____ = _____

Day 7

Goals

To add fractions with a representation of fractions.

Materials

Overhead projector and markers
Learning Sheet 7, one per student
Overhead transparency of Learning Sheet 7
Fraction cards, 2 per student
2 transparencies of fraction cards for the overhead projector

The Instructional Procedures

Give an Advance Organizer

1. **Tell the students what they will be doing and why.**

    Sample dialogue :

    Yesterday we used fraction pieces to add and subtract fractions. You did a good job on your Learning Sheet. Today we're going to use fraction cards to add fractions. It is important to be able to add fractions using various kinds of objects.

Describe and Model Task or Skill

Distribute fraction cards, 2 per student. Place 2 fraction card transparencies side by side on the overhead projector. Introduce fraction cards, which students will use to draw fractions. When you are directed to "set up the problem," that means you are to rename fractions before adding and copy the fractions beside the fractions on the learning sheet. Here's an example:

$$\begin{array}{r} 1\frac{3}{4}=\frac{7}{4} \\ +\ \frac{3}{4}=\frac{3}{4} \\ \hline \\ \underline{\hspace{2cm}}=\underline{\hspace{2cm}} \end{array}$$

Sample dialogue:

These are fraction cards. We can use fraction cards to draw lines that represent fractions. (Point to the fraction 1/2 on the transparency fraction card.) Here is one-half. If I draw a line from the end of the fraction card to this point, I have a line that represents one-half. (Demonstrate the line on the overhead projector.)

If I want to draw two-halves, I draw two lines, each one-half long. (Demonstrate drawing two one-half lines. To find out how long this is, I put my card up to the line and see how long it is. (Demonstrate.) I see that my line is two-halves. What is another name for two-halves? (Elicit the response, "One whole.")

Procedure 1: Teacher asks question and teacher answers question to help students measure lines that represent fractions. Remind students that you will be solving the first problem and thinking aloud as you do it. This should help them to know what to think when they solve their own problems.

Sample dialogue:
Look at Problem 1. (Point to the line.) Here is a line that represents a fraction. The fraction card includes halves, thirds, fourths, fifths, sixths, and eighths. I will use my fraction card to discover how long the line is. (Demonstrate by placing the fraction card alongside the line by Problem 1. The line represents one-fourth.) Is it one-third? No. Is it one-sixth? No. Is it one-fourth? Yes, it is one-fourth. I'll write that on my learning sheet, and you can copy it on your learning sheet. (Model and monitor.)

Procedure 2: Teacher asks questions and students help answer questions to help students measure lines that represent fractions.

Sample dialogue:
Look at Problem 2. There's another line that represents a fraction (one-twelfth). We can use our fraction cards to find out how long it is. Put your fraction card alongside the line. What fraction names the length of the line? Is it one-third? (Elicit the response, "No.") Is it one-sixth? (Elicit the response, "No.") What is the fraction? (Elicit the response, "One-twelfth.") Yes, it is one-twelfth. Let's write that on our learning sheet beside Problem 2. Twelfths are fractions that we haven't used before. You can use your fraction card to find equivalent fractions to twelfths. Equivalent fractions are listed on the fraction cards. (Point to the equivalent fractions on the transparency fraction cards. For example, how many sixths are in four-twelfths? (Elicit the response, "Two-sixths.") (Continue with one or two more examples of finding equivalent fractions to twelfths.)

Procedure 1: Teacher asks question and teacher answers question to help students add fractions.

Sample dialogue:
Look at Problem 3. This is an addition problem. The lines have been drawn already. I'll do this one. First I'll check to see if the denominators are the same. I see that they are, so I can add numerators. The first fraction, one-fourth, is represented by a line. (Point to the line representing one-fourth.) The second fraction is two-fourths , shown here. (Point to the line representing two-fourths.) I can find out much one-fourth and two-fourths are by using my fraction card. (Demonstrate by putting your fraction card alongside the line by Problem 3.) I see that one-fourth plus two-fourths is three-fourths. I'll write that on my learning sheet and you can copy it on yours.

Procedure 2: Teacher asks questions and students help answer questions to help students add fractions.

Sample dialogue:
Look at Problem 4. Here's an addition problem, one and five-eighths plus seven-eighths. What do we check first? (Elicit the response, "Check to see if the denominators are the same.") The denominators are the same, so we will not have to rename before we add. How long do you think this first line is? (Elicit the response, "Five-eighths.") Yes, the line is five-eighths. How long is the second line? (Elicit the response, "Seven-eighths.") Right. It's going to take both of your fraction cards to measure the length of the whole line. I'll show you on the overhead. (Demonstrate with the fraction cards how to measure the total.) Now measure on your learning sheet by Problem 4. What do you get? (Elicit the response, One and four-eighths.") Right, you have one and four-eighths. Write four-eighths in your fraction column on your learning sheet and one in your whole number column. (Model.) Four-eighths can be represented in simpler form. What is the simpler form? (Guide the students to respond, "One-half."). Great. Let's write two and one-half on our learning sheets. (Model and monitor.)

Sample dialogue:
Now it's your turn to do Problem 5, one and three-fourths plus one and one-half. What do you check first? (Elicit the response, "Check to see if the denominators are the same.") The denominators are not the same, so you know that something will have to be renamed. What will you trade in this problem? (Elicit the response, "Rename the

half as fourths.") Yes. What will one and one-half become when you rename? (Elicit the response, "One and two-fourths.") (Set up the problem.) We can look at this problem as having a fractions column and a whole number column. (Point to the columns as you describe them.) Add the fractions column first using your fraction cards. What is the total for fractions? (Elicit the response, "Five-fourths.") Good. Write that in the fraction column of your problem. What is your total for whole numbers? (Elicit the response, "Two.") Write that in the whole number column. Now you have Two and five-fourths. You are not finished with the problem until it is in simplest form. What do you need to do to two and five-fourths to put it in simplest form? (Elicit the response, "Get another whole number out of the five-fourths.") What mixed number do you have now? (Elicit the response, "Three and one-fourth.") Right. Write that number on your learning sheet.

Conduct Guided Practice

1. **Guide the students through Problem 6.** Do not demonstrate the process unless further demonstrations appear necessary.

Sample dialogue:
Look at Problem 6. Find the line that goes with the first fraction and the line that goes with the second fraction. Use your fraction cards to add the fractions. Remember to add the whole numbers that are in the problem.

2. **Guide the students through Problem 7 using the same procedure that you followed with Problem 6.** Do not ask for the answer, however, as this is the first problem scored on the Learning Sheet.

3. **Instruct the students to compute Problem 8 by themselves.** Tell the students to add the fractions and whole numbers, but do *not* guide them through the process. Provide assistance only if needed. Again, do not ask for the answer as this is the second problem to be scored on the Learning Sheet.

Conduct Independent Practice

1. **Instruct the students to solve Problems 9-14 independently.**

Sample dialogue:
Now do Problems 9 through 14 on your Learning Sheet. When you finish Problem 14, stop. What are you going to do?

2. **Repeat directions if necessary.**
3. **Tell the students to begin.**
4. **Circulate and monitor the students' work while they solve the problems.**

Describe and Model/Guided Practice LS 7 St. #_____

1. ________ **2.** ___________

3.

$$\begin{array}{r} \frac{1}{4} \\ +\frac{2}{4} \\ \hline \end{array}$$

_____

4.

$$\begin{array}{r} 1\frac{5}{8} \\ +\ \frac{7}{8} \\ \hline \end{array}$$

_____=_____

5.

$$\begin{array}{r} 1\frac{3}{4} \\ +1\frac{1}{2} \\ \hline \end{array}$$

_____=_____

6. $\dfrac{2}{6}$

$+\dfrac{5}{6}$

_____

_____=______

7. $1\dfrac{4}{8}$

$+\ \dfrac{3}{4}$

_____

_____=______

8. $1\dfrac{2}{12}$

$+1\dfrac{1}{2}$

_____

_____=______

Independent Practice

9. $\dfrac{3}{8}$

$+\dfrac{2}{4}$

_____

_____=_____

10. $1\frac{1}{6}$

$+1\frac{2}{3}$

_____

_____=_____

11. $\frac{2}{3}$

$+\frac{1}{2}$

_____

_____=_____

12. $1\frac{2}{8}$

$+\ \frac{1}{2}$

_____

_____=_____

13. $\frac{1}{12}$

$+\frac{1}{4}$

_____

_____=_____

14. $1\frac{3}{12}$

$+\ \frac{1}{3}$

_____

_____=_____

Day 12

Goals

To promote the students' ability to respond from memory to fraction addition and subtraction facts.

To practice rules that will aid in fraction addition and subtraction.

Materials

Overhead projector and markers
Fraction Shortcuts Practice Sheet
Fraction Shortcuts Sheet
Overhead transparency of *Fraction Shortcuts Sheet*
Overhead transparency of *Fraction Shortcuts Practice Sheet*

Fraction cards, 2 per student
2 transparencies of fraction cards for the overhead projector
(The fraction cards may be needed to work if students have trouble with the concepts.)

The Instructional Procedures
Give an Advance Organizer

1. Tell the students what they will be doing and why.

Sample dialogue:
In our last lesson, you practiced rules for adding and subtracting fractions and the "DRAW" Strategy. (Ask students to name and explain each rule. Then quickly review each step of "DRAW.")

Today, we'll continue to use "DRAW" to solve problems. We'll also learn some fraction shortcuts. By knowing these shortcuts really well, you'll be able to solve problems even faster because you'll know the answers to problems without drawing lines.

Demonstrate and Model the Task or Skill/Conduct Guided Practice

1. Give each student a Fraction Shortcuts Sheet.

2. Read and explain the first and second shortcut on the sheet.

Sample dialogue:
This sheet tells us some shortcuts about how to add and subtract fractions, find fraction equivalents, and make mixed numbers into fractions and fractions into mixed numbers. The first shortcut is called Langlin', Ranglin', and Danglin'. The **L in Langlin'**stands for multiplying to left. The **R in Ranglin'** stands for multiplying to the right. You don't want your

denominator to be left **danglin',** do you? The **D in Danglin'** reminds you to multiply your denominators also. You use this shortcut when you are adding or subtracting fractions. First you cross-multiply from the left to the right and write the answer above one fraction. (Model this on the transparency.) Next, cross-multiply the other way and write the answer above the other fraction. (Model this on the transparency.) If you are adding fractions, you add these top numbers to get the numerator. If you are subtracting fractions, you get the difference between these two numbers to get the denominator. Finally, the denominator of the answer, you multiply the denominators. (Model this on the transparency.) You have an answer now, but you may still need to put the fraction in simplest form.

3. **Read and explain the third shortcut on the sheet.**

   Sample dialogue:
   The second shortcut shows you how to change mixed numbers to fractions. You start at the bottom with the denominator, which you multiply by the whole number. (Model this on the transparency.) In this example, you have 20. Next, you add the numerator to your total, which gives you 21. This is the numerator of your fraction. You keep the same denominator, so your answer is twenty-one-fourths.

4. **Read and explain the fourth shortcut on the sheet.**

   Sample dialogue:
   This shortcut shows you how to convert a fraction to a mixed number. To do this, you divide the denominator into the numerator. In this case, you divide 3 into 17. Three goes into seventeen five times, so five is your whole number. The remainder, or two, is the numerator of the fraction, and you keep the same denominator. The final answer is 5 and two-thirds.

5. **Read and explain the fifth shortcut on the sheet.**

   Sample dialogue:
   This shortcut shows you how to generate equivalent fractions. When you multiply or divide the numerator and the denominator by the same number, you generate fractions that are equivalent. For example, with the fraction three-fourths, if you multiply three over three, you have 9 in the numerator and 12 in the denominator for a fraction of nine-twelfths, which is equivalent to three-fourths.

6. **Give each student a copy of the *Fraction Shortcuts Practice Sheet.***

7. **Demonstrate how to compute Problem 1.**

Sample dialogue:
Look at Problem 1 on your Practice Sheet. Let's do this one together. It asks us to solve the addition problem, two-thirds plus three-fourths. We'll use ranglin', langlin', and danglin' to solve the problem. What do we multiply first when we rangle to the right? (Elicit the response, "Three times three equals nine.") Yes, write that number above the second fraction. What do we multiply next when we langle to the left? (Elicit the response, "Four times two equals eight.") Yes, write that number above the first fraction. We now add those two numbers above the fractions to get our numerators. What will the numerator be? (Elicit the response, "Seventeen.") When we dangle, we multiply the denominators to get a new denominator. What will our new denominator be? (Elicit the response, "Twelfths.") That's right. What is the new fraction? (Elicit the response, "Seventeen-twelfths.") Yes, let's write that on our learning sheets. (Model and monitor.) Are we finished with this problem? (Elicit the response, "No, because it is not in simplest form.") What rule will you use to rename the fraction in simplest form? (Elicit the response, "When the numerator is larger than the denominator, group any wholes together.") What is your answer? (Elicit the response, "One and five-twelfths.") Correct. Let's write that on our learning sheets. (Monitor to see that students write the answers on their learning sheets.)

8. **Demonstrate how to compute Problem 2.**

Sample dialogue:
Look at Problem 2 on your Practice Sheet. Let's do this one together. It asks us to solve the subtraction problem, two-sixths minus two-eighths. We'll use ranglin', langlin', and danglin' to solve the problem. What do we multiply first when we rangle to the right? (Elicit the response, "Six times two equals twelve.") Yes, write that number above the second fraction. What do we multiply next when we langle to the left? (Elicit the response, "Eight times two equals sixteen.") Yes, write that number above the first fraction. We now find the difference between those two numbers above the fractions to get our numerators. What will the numerator be? (Elicit the response, "Four.") When we dangle, we multiply the denominators to get a new denominator. What will our new denominator be? (Elicit the response,

"Forty-eight.") That's right. What is the new fraction? (Elicit the response, "Four forty-eighths.") Yes, let's write that on our learning sheets. (Model and monitor.) Are we finished with this problem? (Elicit the response, "No, because it is not in simplest form.") What is your answer? (Elicit the response, "one-twelfth.") Correct. Let's write that on our learning sheets. (Monitor to see that students write the answers on their learning sheets.)

9. **Demonstrate how to compute Problem 3.**

Sample dialogue:
Let's do Problem 3 together. We will change the mixed number, one and five-sixths to a mixed number. What do we do first? (Elicit the response, "Multiply the denominator, six, by the whole number, one.") That's right. Then what do you do? (Elicit the response, "Add the numerator, four, to make ten.") Right again. This number is your numerator. What is your denominator? (Elicit the response, "Six.") So, what is your answer? (Elicit the response, "Eleven-sixths.") Very good. Let's write that on our learning sheets. (Model and monitor.)

10. **Demonstrate how to compute Problem 4.**

Sample dialogue:
Let's do Problem 4 together. Now we'll change a fraction, fourteen-thirds, to a mixed number. What do we do first? (Elicit the response, "Divide the numerator by the denominator to get the whole number.") Right. What will the whole number be? (Elicit the response.)

11. **Demonstrate how to compute Problem 5.**

Sample dialogue:
Let's do Problem 5 together. The problem says, "Three-eighths equals blank sixteenths." To solve this problem, we remember our shortcut about multiplying the numerator and denominator by the same number. In this problem, we must decide what 8 was multiplied by to get 16 and multiply tha same number by the numerator, three. This will give us our missing numerator. What number times 8 is 16? (Elicit the response, "Two.") That's correct. The denominator, 8, was multiplied by 2 to get 16. To form an equivalent fraction, we multiply the numerator by the same number as the denominator. When we multiply 3 times 2, what do we get? (Elicit the response, "Six.") Correct. Six goes in the blank for Problem 5.

Instruct students to complete the *Fraction Shortcuts Practice Sheet,*, using one of the shortcuts just reviewed. Provide assistance as needed by demonstrating the rules with examples, but do not provide answers to the problems. Items 6 through 15 will be scored as independent practice.

12. **When all the students have completed the *Fraction Shortcuts Practice Sheet,* orally provide the correct answers.** Have the students check their own papers and chart their progress.

Fraction Shortcuts LS 12a St. #_____

Langlin', Ranglin', Danglin' for Addition

$\frac{1}{5} + \frac{2}{3} = \frac{13}{15}$

1. Cross-multiply, add, and multiply denominators.
2. Put fraction in simplest form.

Langlin', Ranglin', Danglin' for Subtraction

$\frac{1}{5} - \frac{2}{3} = \frac{7}{15}$

1. Cross-multiply, subtract, and multiply denominators.
2. Put fraction in simplest form.

Mixed number to fraction

$5\frac{1}{4} = \frac{21}{4}$

1. Multiply denominator times whole number.
2. Add numerator.
3. Keep original denominator.

Fraction to mixed number

$\frac{17}{3} = 5\frac{2}{3}$

1. Divide numerator by denominator to get whole number.
2. Remainder is numerator of fraction.
3. Keep original denominator.

Equivalent fractions

$\frac{3}{4} \times \frac{2}{2} =$ $\frac{3}{4} \times \frac{4}{4} =$

1. Multiply both numerator and denominator by the same number.

Fraction Shortcuts Practice LS 12b St. #_____

1. $\frac{2}{3} + \frac{3}{4} =$ ____

2. $\frac{2}{6} - \frac{2}{8} =$ ____

3. $1\frac{5}{6} =$ ____

4. $\frac{14}{3} =$ ____

5. $\frac{3}{8} = \frac{\_}{16}$

Independent Practice

Add.

6. $\frac{3}{8} + \frac{1}{2} =$ ____

7. $\frac{3}{6} + \frac{3}{2} =$ ____

Change the mixed number into a fraction.

7. $3\frac{2}{4} =$ ____

8. $5\frac{3}{8} =$ ____

Change the fraction into a mixed number.

9. $\frac{9}{2} =$ ____

10. $\frac{15}{7} =$ ____

Subtract.

11. $\frac{5}{10} - \frac{1}{4} =$ ____

12. $\frac{6}{8} - \frac{3}{4} =$ ____

Make the fractions equivalent.

14. $\frac{7}{8} = \frac{\_}{24}$

15. $\frac{2}{6} = \frac{6}{\_}$

APPENDIX F
TEST SEQUENCE

Order of Tests **Teacher** ____________________

id#	Post 1	Post 2	Post 3		id#	Post 1	Post 2	Post 3

APPENDIX G
STUDENT FEEDBACK ON FRACTIONS

Student Feedback on Fractions

When you learned about fractions this spring, you used fraction pieces, fraction cards, and rules and shortcuts to solve fraction problems. I'd like to know how you feel about this way to learn fractions.

Please answer these questions about the fraction unit. Thank you for your ideas.

1. What did you like about the fraction pieces?

2. What did you dislike about the fraction pieces?

3. What did you like about the fraction card?

4. What did you dislike about the fraction card?

5. What did you like about the fraction rules and shortcuts?

6. What did you dislike about the fraction rules and shortcuts?

7. If you could make changes in the fraction instruction you had, what would they be?

8. What did you think about the learning sheets?

9. Do you think you learned fractions any better than if you had been using your math book? Why or why not?

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BIOGRAPHICAL SKETCH

LuAnn Jordan was born in Hertford, North Carolina, on October 7, 1961. She graduated from Albemarle Academy, Elizabeth City, North Carolina, in 1979. LuAnn received an A.A. degree in pre-elementary education from College of the Albemarle in 1981; a B.S. degree in special education-mental retardation from East Carolina University, Greenville, North Carolina, in 1983; and a Master of Education degree in special education-learning disabilities also from ECU in 1988. In 1990, LuAnn began coursework in administration and supervision at East Carolina University.

LuAnn taught students with mild mental disabilities in grades 1 through 3 at Sam D. Bundy Elementary School, Farmville, North Carolina, from 1983 to 1991. While teaching in Farmville, LuAnn was honored as North Carolina Association of Teacher Educators Outstanding Beginning Teacher and Pitt County Teacher of the Year.

While completing her doctoral studies at the University of Florida, LuAnn has served as graduate teaching assistant, graduate research assistant, and practicum and intern supervisor in the Department of Special Education, College of Education. During this time, she also served as educational diagnostician and later as research associate for the Multidisciplinary Diagnostic and Training Program in the College of Education.

In addition to her training at the University of Florida, LuAnn also received training in the Strategies Intervention Model in 1993 at the University of Kansas. This program is directed by Dr. Don Deshler, Dr. Jean Schumaker, Dr. Keith Lenz, and Dr. Fran Clark.

LuAnn has served as secretary of the Special Education Association of Graduate Students in the Department of Special Education (1992-1993). She is a member of Delta Kappa Gamma Society International, Eta State, Beta Alpha Chapter, where she served as secretary (1989-1991). LuAnn served as Pitt County president of CEC (1990-1991). She is a member of Phi Delta Kappa, the ARC, the Learning Disabilities Association (LDA) of America, and Kappa Delta Pi and Phi Kappa Phi honor societies. LuAnn plans to continue her work in teacher preparation and to continue her research in interventions for students with mild disabilities.

I certify that I have read this study and that in my opinion it conforms to accepted standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Cecil D. Mercer

Cecil D. Mercer, Chairperson
Professor of Special Education

I certify that I have read this study and that in my opinion it conforms to accepted standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Donald H Bernard

Donald H. Bernard
Associate Professor of Instruction and Curriculum

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Cynthia Griffin
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M. David Miller

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Associate Professor of Foundations of Education

I certify that I have read this study and that in my opinion it conforms to accepted standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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This dissertation was submitted to the Graduate Faculty of the College of Education and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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